# Differentiation

## 1 Chain Rule

You met this at Higher!

If $k(x) = f(g(x))$

then $k'(x) = f'(g(x)) \bullet g'(x)$

### Example

Differentiate $y = sin^2(5x - 1)$ with respect to $x$.

$$y = (\sin(5x-1))^2$$

$$\frac{dy}{dx} = 2\left(\sin(5x-1)\right) \bullet \cos\left(5x-1\right) \bullet 5$$

$$= 10\sin(5x-1) \bullet \cos(5x-1)$$

$$= \underline{\underline{5\sin 2(5x-1)}}$$

* Since $sin2a = 2sinacosa$

## 2 Product Rule

$y = x^3 sinx$ is the product of two functions $u$ and $v$.

$$y = x^3 sinx$$

where $u = x^3$ and $v = sinx$

In general, if $y = uv$ , where $u$ and $v$ are functions of $x$ :

$$f'(x) = u'v + v'u$$

## Example 1

$y = x^3 sinx$ where $u = x^3$ $\quad u' = 3x^2$

$v = sinx$ $\quad v' = cosx$

$$\frac{dy}{dx} = u'v + v'u$$

$$= 3x^2 \bullet sinx + cosx \bullet x^3$$

$$= \underline{\underline{3x^2 sinx + x^3 cosx}}$$

## Example 2

$y = x\cos 2x$ where $u = x$ $u' = 1$

$v = \cos 2x$ $v' = -2\sin 2x$

$$\frac{dy}{dx} = u'v + v'u$$

$$= \cos 2x - 2\sin 2x \bullet x$$

$$= \underline{\underline{\cos 2x - 2x\sin 2x}}$$

## Example 3

$y = x^2(x-1)^{10}$ where $u = x^2$ $u' = 2x$

$v = (x-1)^{10}$ $v' = 10(x-1)^9$

$$\frac{dy}{dx} = u'v + v'u$$

$$= 2x \bullet (x-1)^{10} + 10(x-1)^9 \bullet x^2$$

$$= 2x(x-1)^{10} + 10x^2\left(x-1\right)^9$$

$$= \underline{\underline{2x(x-1)^9\left(6x-1\right)}}$$

## Example 4

$y = sin^3 x cos^2 x$

$y = (sinx)^3 (cosx)^2$ where $u = (sinx)^3$ $u' = 3cosxsin^2 x$

$v = (cosx)^2$ $v' = -2cosxsinx$

$$\frac{dy}{dx} = u'v + v'u$$

$$= 3cosxsin^2 x \bullet cos^2 x - 2cosxsinx \bullet sin^3 x$$

$$= \underline{\underline{3cos^3 xsin^2 x - 2cosxsin^4 x}}$$

## Example 5

$y = x(2x+5)^5$ where $u = x$ $u' = 1$

$v = (2x+5)^5$ $v' = 10(2x+5)^4$

$$\frac{dy}{dx} = u'v + v'u$$

$$= (2x+5)^5 + 10(2x+5)^4 \bullet x$$

$$= (2x+5)^5 + 10x(2x+5)^4$$

$$= \underline{\underline{(2x+5)^4(12x+5)}}$$

# Example 6

Find the equation of the tangent to the curve $y = x\sqrt{x+1}$ at the point where $x = 3$ .

$y = x\sqrt{x+1}$ where

$$u = x \qquad u' = 1$$

$$v = (x+1)^{\frac{1}{2}} \qquad v' = \tfrac{1}{2}(x+1)^{-\frac{1}{2}} = \frac{1}{2(x+1)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = u'v + v'u$$

$$= (x+1)^{\frac{1}{2}} + \frac{1}{2(x+1)^{\frac{1}{2}}} \bullet x$$

$$= \frac{2(x+1)}{2(x+1)^{\frac{1}{2}}} + \frac{x}{2(x+1)^{\frac{1}{2}}}$$

$$= \underline{\underline{\frac{3x+2}{2\sqrt{x+1}}}}$$

Gradient at $x = 3$: $$m = \frac{3(3)+2}{2\sqrt{3+1}} = \frac{11}{4}$$

Point on line when $x = 3$: $y = 3\sqrt{3+1} = 6$

Equation of line: $$y - b = m(x-a)$$

$$y - 6 = \tfrac{11}{4}(x-3)$$

So, $\underline{\underline{11x - 4y - 9 = 0}}$

## 3 Quotient Rule

If $y = \frac{u}{v}$, where $u$ and $v$ are functions of $x$ then:

$$f'(x) = \frac{u'v - v'u}{v^2}$$

## Example

$y = \frac{2+3x}{1-2x}$ where $u = 2 + 3x$ $\quad u' = 3$

$v = 1 - 2x$ $\quad v' = -2$

$$\frac{dy}{dx} = \frac{u'v - v'u}{v^2}$$

$$= \frac{3(1-2x) - (-2)(2+3x)}{(1-2x)^2}$$

$$= \frac{7}{(1-2x)^2}$$