Proofs

Direct Proof

Example 1

This is a simple, *direct proof* that the sum of two integers is itself an even number.

Proof: Consider two even integers x and y.

Since they are even, they can be written as $x = 2a$ and $y = 2b$ respectively for integers a and b.

Then the sum $x + y = 2a + 2b = 2(a + b)$

From this, it is clear that $x + y =$ has 2 as a factor and therefore is even, so the sum of any two even integers is even.

Example 2

Given that r and s are rational numbers, show that $r + s$ is rational.

Proof: Since r and s are rational, we can write:

$$r = \frac{p}{q} \quad \text{and } s = \frac{m}{n} \quad m, n, p, q \in \mathbb{Z}$$

Then $r + s = \frac{p}{q} + \frac{m}{n}$

$$= \frac{pn+qm}{qn}$$

Since m, n, p, q are integers, $pn + qm$ and qn are also integers.

Thus, by the calculation above, $r + s$ is the quotient of two integers, and is therefore a rational number.

Example 3

Here's an example of a proof that is really just a calculation.

Given the trig identity $\sin(x + y) = \sin x \cos y + \cos x \sin y$, prove the identity: $\sin(2x) = 2\sin x \cos x$

Proof: $\sin(2x) = \sin(x + x)$
$= \sin x \cos x + \cos x \sin x$
$= 2\sin x \cos x$

2013 Q12

Let n be a natural number.
For each of the following statements, decide whether it is true or false.
If true, give a proof; if false, give a counterexample.

A If n is a multiple of 9 then so is n^2.

B If n^2 is a multiple of 9 then so is n. **4**

	Let n be a natural number. For each of the following statements, decide whether it is true or false. If true, give a proof, if false, give a counterexample. **A** **If n is a multiple of 9 then so is n^2.** **B** **If n^2 is a multiple of 9 then so is n.**	**4**	
A	Suppose $n = 9m$ for some natural number [positive integer], m.		•[1] Generalisation, using *different* letter.[3, 6]
	Then $n^2 = 81m^2 = 9(9m^2)$		•[2] Correct multiplication *and* 9 extracted as a factor.
	Hence n^2 is a multiple of 9, so **A** is **true**.		•[3] Conclusion of proof *and* state A true.[1]
B	**False.** Accept any valid counterexample: $n = 3, 6, 12, 15, 21$ etc		•[4] Valid counterexample *and* conclusion.[5]

2010 Q8

(*a*) Prove that the product of two odd integers is odd. **2**

(*b*) Let p be an odd integer. Use the result of (*a*) to prove by induction that p^n is odd for all positive integers n. **4**

(a) Write the odd integers as: $2n + 1$ and $2m + 1$ where n and m are integers.	**1M**	for unconnected odd integers
Then $(2n+1)(2m+1) = 4nm + 2n + 2m + 1$ $= 2(2nm + n + m) + 1$	**1**	demonstrating clearly
which is odd.		
(b) Let $n = 1$, $p^1 = p$ which is given as odd.	**1**	
Assume p^k is odd and consider p^{k+1}.	**1M**	
$p^{k+1} = p^k \times p$	**1**	
Since p^k is assumed to be odd and p is odd, p^{k+1} is the product of two odd integers is therefore odd.	**1**	for a valid explanation from a previous correct argument
Thus p^{n+1} is an odd integer for all n if p is an odd integer.		

2008 Q11

For each of the following statements, decide whether it is true or false and prove your conclusion.

A For all natural numbers m, if m^2 is divisible by 4 then m is divisible by 4.

B The cube of any odd integer p plus the square of any even integer q is always odd. **5**

(a)	Counter example $m = 2$. So statement is false.	**1,1**
(b)	Let the numbers be $2n + 1$ and $2m$ then	**1M**
	$(2n+1)^3 + (2m)^2 = 8n^3 + 12n^2 + 6n + 1 + 4m^2$	**1**
	$= 2(4n^3 + 6n^2 + 3n + 2m^2) + 1$	**1**
	which is odd.	
	OR	
	Proving algebraically that either the cube of an odd number is odd or the square of an even number is even.	**1**
	Odd cubed is odd and even squared is even.	**1**
	So the sum of them is odd.	**1**