Unit 2
Further Integration

FORMULAE LIST

Standard derivatives

$f(x)$	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\tan x$	$\sec^2 x$
$\ln x,\ x>0$	$\frac{1}{x}$
e^x	e^x

Standard integrals

$f(x)$	$\int f(x)\,dx$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)+c$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax)+c$
e^{ax}	$\frac{1}{a}e^{ax}+c$

Ex 1

$$\int \frac{1}{\sqrt{9-x^2}}dx$$

$$= \int \frac{1}{\sqrt{3^2-x^2}}dx$$

$$= \sin^{-1}\left(\frac{x}{3}\right)+C$$

Ex 2

$$\int \frac{1}{9+x^2}dx$$

$$= \int \frac{1}{3^2+x^2}dx$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+C$$

Ex 3

$$\int \frac{1}{\sqrt{5-x^2}}dx$$

$$= \int \frac{1}{\sqrt{\left(\sqrt{5}\right)^2-x^2}}dx$$

$$= \sin^{-1}\left(\frac{x}{\sqrt{5}}\right)+C$$

Ex 4

$$\int \frac{4}{x^2+36}dx = 4\int \frac{1}{6^2+x^2}dx$$

$$= 4.\frac{1}{6}\tan^{-1}\left(\frac{x}{6}\right)+C$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{x}{6}\right)+C$$

Ex 5 $\int \frac{1}{\sqrt{25-9x^2}}dx = \int \frac{1}{\sqrt{5^2-(3x)^2}}dx$

$$= \frac{1}{3}\sin^{-1}\left(\frac{3x}{5}\right)+C$$

Note that since x is replaced by $3x$ in the standard integral, the factor $\frac{1}{3}$ must be included to compensate.

Ex 6 $\int \frac{1}{9+4x^2}dx = \int \frac{1}{3^2+(2x)^2}dx$

$$= \frac{1}{2}.\frac{1}{3}\tan^{-1}\left(\frac{2x}{3}\right)+C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)+C$$

Note that since x is replaced by $2x$ in the standard integral, the factor $\frac{1}{2}$ must be included to compensate.

Ex 7 $\int_{\frac{3}{2}}^{3} \frac{1}{\sqrt{9-x^2}}dx = \int_{\frac{3}{2}}^{3} \frac{1}{\sqrt{3^2-x^2}}dx$

$$= \left[\sin^{-1}\left(\frac{x}{3}\right)\right]_{\frac{3}{2}}^{3}$$

$$= \sin^{-1}(1)-\sin^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{2}-\frac{\pi}{6}$$

$$= \frac{3\pi}{6}-\frac{\pi}{6}$$

$$= \frac{2\pi}{6} = \frac{\pi}{3}$$

Ex 8 $\int_{-2}^{2} \frac{1}{4+x^2}dx = \int_{-2}^{2} \frac{1}{2^2+x^2}dx$

$$= \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$$

$$= \frac{1}{2}\left[\left(\tan^{-1}(1)\right)-\left(\tan^{-1}(-1)\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{2}\left[\frac{2\pi}{4}\right] = \frac{\pi}{4}$$

Ex 9 Using the Standard Integral:

$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

find,

$$\int \frac{1}{x^2-6x+13}\,dx = \int \frac{1}{(x-3)^2+4}\,dx$$

(by completing the square)

$$= \int \frac{1}{(x-3)^2+2^2}\,dx$$

$$= \int \frac{1}{2^2+(x-3)^2}\,dx$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right)+C$$

This question could be given as:

Use the substitution $u = x-3$

to find the indefinite integral $\int \frac{1}{x^2-6x+13}\,dx$

$$\int \frac{1}{x^2-6x+13}\,dx \qquad u = x-3$$

$$du = dx$$

Now $x^2-6x+13$ must be expressed in terms of u.

$u = x-3 \quad \Rightarrow \quad x-3 = u$

$\quad \Rightarrow \quad x = u+3$

$$x^2-6x+13 = (u+3)^2-6(u+3)+13$$

$$= u^2+6u+9-6u-18+13$$

$$= u^2+4$$

$$\int \frac{1}{x^2-6x+13}\,dx = \int \frac{1}{u^2+4}\,du$$

$$= \int \frac{1}{2^2+u^2}\,du$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)+C$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right)+C$$

Ex 10

Use the substitution $u = 2\sin x$ to evaluate $\int_0^{\frac{\pi}{6}} \frac{\cos x}{1+4\sin^2 x} dx$

$u = 2sinx$

$du = 2cosxdx$

$$1+4\sin^2 x = 1+(2\sin x)^2$$
$$= 1+u^2$$

$$\int_0^{\frac{\pi}{6}} \frac{\cos x}{1+4\sin^2 x} dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} \frac{du}{1+u^2}$$

New Limits

When $x = 0$: $u = 2\sin 0 = 2\times 0 = 0$

When $x = \frac{\pi}{6}$: $u = 2\sin\frac{\pi}{6} = 2\times\frac{1}{2} = 1$

$$= \frac{1}{2}\int_0^1 \frac{1}{1+u^2} du$$

$$= \frac{1}{2}\left[\tan^{-1}(u)\right]_0^1$$

$$= \frac{1}{2}\left[\left(\tan^{-1}(1)\right) - \left(\tan^{-1}(0)\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4}\right] \quad = \frac{\pi}{8}$$