Complex Numbers

Complex Numbers

Since $x^2 \geq 0$ for every real number x, the equation $x^2 = -1$ has no real solutions.

To deal with this problem, mathematicians of the eighteenth century introduced the "imaginary" number,

$$i = \sqrt{-1}$$

which they assumed had the property

$$i^2 = \left(\sqrt{-1}\right)^2 = -1$$

but which otherwise could be treated like an ordinary number.

Expressions of the form

$$a + bi$$

where a and b are real; numbers, were called "complex numbers" and these were manipulated according to the standard rules for arithmetic with the added property that $i^2 = -1$.

By the beginning of the nineteenth century it was recognised that a complex number could be regarded as an alternative symbol for the order pair

$$(a, b)$$

DEFINITION

A *complex number* is an ordered pair of real numbers, denoted either by (a, b) or by by $a + bi$, where $i^2 = -1$.

Note

$$i = \sqrt{-1}$$

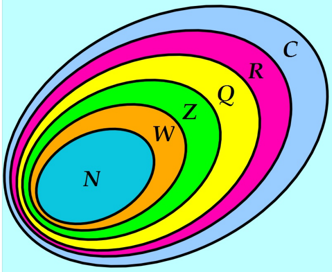
$$i^2 = -1$$

$$i^3 = i^2 \times i = -i$$

$$i^4 = i^2 \times i^2 = 1$$

Some examples of complex numbers in both notations are as follows:

ORDERED PAIR	EQUIVALENT
(3, 4)	3 + 4i
(-1, 2)	-1 + 2i
(0, 1)	i
(2, 0)	2
(4, -2)	4 - 2i



N ~ Natural numbers 1, 2, 3, ...

W ~ Whole numbers 0, 1, 2, 3, ...

Z ~ Integers ....-3, -2, -1, 0, 1, 2, 3, ...

Q ~ Rational numbers
all fractions of the form $\frac{m}{n}$

R ~ Real numbers
all numbers rational and irrational
$-\sqrt{2},\ -1,\ e,\ \pi,\ 27.3$

C ~ Complex numbers, $z = a + b$i

REAL PART ***IMAGINARY PART***

The Complex Plane

Sometimes it is convenient to use a single letter, such as z, to denote a complex number.

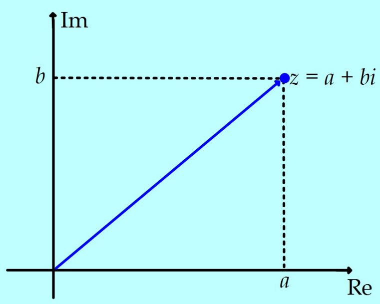
Thus we might write

$$z = a + bi$$

'a' is the REAL PART of z or Re(z)
'b' is the IMAGINARY PART of z or Im(z)

Thus Re(4 - 3i) = 4 and Im(4 - 3i) = -3

The resulting plot is called an **Argand Diagram**.



Exercise

$z = 1 + i$ **Re**z $= 1$

Imz $= 1$

$z = -\frac{5}{2} - \sqrt{3}i$ **Re**z $= {}^{5}/_{2}$

Imz $= -\sqrt{3}$

$z = 10$ **Re**z $= 10$

Imz $= 0$

$z = i$ **Re**z $= 0$

Imz $= 1$

Exercise

$(2i)^2$
$= 2^2 i^2$
$= \underline{\underline{-4}}$

$(-3i)^2$
$= 9i^2$
$= \underline{\underline{-9}}$

$\left(\sqrt{3}i\right)^2$
$= 3i^2$
$= \underline{\underline{-3}}$

$\left(-\sqrt{\pi} i\right)^2$
$= \pi i^2$
$= \underline{\underline{-\pi}}$

$i = \sqrt{-1}$
$i^2 = -1$

Operations on Complex Numbers

Equating Complex Numbers

$w = a + bi$
$z = c + di$

Suppose w=z
$a + bi = c + di \Leftrightarrow a = c$ (REAL PART) and $b = d$ (IMAG PART)

Two complex numbers are equal if and only if (iff) both the real and imaginary parts are equal.

Example

$x + 2yi = 3 + (x + 1)i$ where x, y R

$\underline{\underline{x = 3}}$

$2y = x + 1$
$2y = 3 + 1$
$\underline{\underline{y = 2}}$

Operations on Complex Numbers

Solving Equations *(will return to this later)*

Solve $z^2 - 2z + 5 = 0$

$z = \frac{2 \pm \sqrt{4-20}}{2}$

$= \frac{2 \pm \sqrt{-16}}{2}$ $\qquad \sqrt{-16} = \sqrt{16}\sqrt{-1} = 4i$

$= \frac{2 \pm 4i}{2}$

$= \underline{\underline{1 \pm 2i}}$

Operations on Complex Numbers

Addition, Subtraction, Multiplication

- Carry out operations as usual remembering $i^2 = -1$
- Collect together real and imaginary parts
- For division, rationalise the denominator by multiplying by the complex conjugate

Example

Let $z = 4 + 3i \qquad w = 1 - 2i$

ADDITION

$z + w = (4 + 3i) + (1 - 2i)$
$= 4 + 1 + 3i - 2i$
$= \underline{\underline{5 + i}}$

SUBTRACTION

$z - w = (4 + 3i) - (1 - 2i)$
$= 4 - 1 + 3i + 2i$
$= \underline{\underline{3 + 5i}}$

Operations on Complex Numbers

Example

Let $z = 4 + 3i \qquad w = 1 - 2i$

MULTIPLICATION

$zw = (4 + 3i)(1 - 2i)$
$= 4 - 8i + 3i - 6i^2$
$= 4 - 5i + 6$
$= \underline{\underline{10 - 5i}}$

Operations on Complex Numbers

Example

Let $z = 4 + 3i \qquad w = 1 - 2i$

BINOMIAL EXPANSION

$z^4 =$