Matrices

1 A Matrix

A matrix is an array of numbers:

$$\begin{pmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{pmatrix}$$

This one has 2 Rows and 3 Columns
(i.e. a 2×3 matrix)

We talk about one **matrix**, or several **matrices**.

2 Equal Matrices

Two matrices are equal if and only if they are of the same number and all corresponding entries are equal.

e.g. $\begin{pmatrix} 3 & 7 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 1 & 2 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} 3 & 7 \\ 1 & 2 \end{pmatrix} \neq \begin{pmatrix} 3 & 0 & 7 \\ 1 & 0 & 2 \end{pmatrix}$

3 Transposing

To "transpose" a matrix, swap the rows and columns. The resulting matrix is denoted by A^T or A':

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \\ 0 & -6 \end{pmatrix} \qquad A' = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 4 & -6 \end{pmatrix}$$

4 Notation

A matrix is usually shown by a **capital letter** (such as A, or B)

Each entry (or "element") is shown by a **lower case letter** with a "subscript" or **row, column**:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix}$$

To remember that rows come before columns use the word **"arc"**:

$$a_{r,c}$$

5 Adding/ Subtracting

To add two matrices: add the numbers in the matching positions:

$$\begin{pmatrix} 3 & 8 \\ 4 & 6 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 1 & -9 \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ 5 & -3 \end{pmatrix}$$

These are the calculations:

$3 + 4 = 7 \qquad 8 + 0 = 8$

$4 + 1 = 5 \qquad 6 - 9 = -3$

The two matrices must be the same size.

6 Negative

The negative of a matrix is also simple:

$$-\begin{pmatrix} 2 & -4 \\ 7 & 10 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ -7 & -10 \end{pmatrix}$$

These are the calculations:

-(2) = -2 -(-4) = +4

-(7) = -7 -(10) = -10

7 Multiply by a Constant (Scalar)

We can multiply a matrix by some value:

$$2\begin{pmatrix} 3 & 5 \\ 6 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 10 \\ 12 & -2 \end{pmatrix}$$

$$k\begin{pmatrix} 2 & 1 & 3 \\ 7 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 2k & k & 3k \\ 7k & 0 & 5k \end{pmatrix}$$

We call the constant a **scalar**, so officially this is called "scalar multiplication".

Quick Question:

Matrices A and B are defined by $A = \begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix}$

Find the values of p and x if $B = 3A'$.

8 Multiplying by Another Matrix

To multiply a matrix **by another matrix** you need to do the "dot product" of rows and columns... what does that mean?

Let me show you with an example:

To work out the answer for the **1st row** and **1st column:**

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} = \begin{pmatrix} 58 & p_{1,2} \\ p_{2,1} & p_{2,2} \end{pmatrix}$$

$$(1, 2, 3) \cdot (7, 9, 11) = 1\times7 + 2\times9 + 3\times11 = 58$$

The final solution is:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} = \begin{pmatrix} 58 & 64 \\ 139 & 154 \end{pmatrix}$$

Note/

For Matrix Multiplication to work, the number of **columns** of the first matrix must be equal to the number of **rows** of the second matrix.

The matrices are said to be **confrontable**.

$$A = (m \times n) \times (n \times p) = m \times p$$

Further Examples

(i) $$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 2\times 1 + 1\times 5 \\ 0\times 1 + 3\times 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 15 \end{pmatrix}$$

(ii) $$\begin{pmatrix} 3 & 5 \\ 0 & 11 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3\times 1 + 5\times 3 & 3\times 2 + 5\times 0 \\ 0\times 1 + 3\times 5 & 0\times 2 + 11\times 0 \end{pmatrix}$$

$$= \begin{pmatrix} 18 & 6 \\ 33 & 0 \end{pmatrix}$$

Leave room for another two sections.

Why do it this way?

Example: The local shop sells 3 types of pies.

- Beef pies cost **$3** each
- Chicken pies cost **$4** each
- Vegetable pies cost **$2** each

And this is how many they sold in 4 days:

	Mon	Tue	Wed	Thu
Beef	**13**	**9**	**7**	**15**
Chicken	**8**	**7**	**4**	**6**
Vegetable	**6**	**4**	**0**	**3**

Now think about this ... the **value of sales** for Monday is calculated this way:

→ Beef pie value + Chicken pie value + Vegetable pie value

→ $\$3\times13 + \$4\times8 + \$2\times6 = \$83$

So it is, in fact, the "dot product" of prices and how many were sold:

$$(\$3, \$4, \$2) \bullet (13, 8, 6) = \$3\times13 + \$4\times8 + \$2\times6 = \$83$$

We **match** the price to how many sold, **multiply** each, then **sum** the result.
So it is important to match each price to each quantity.

And here is the full result in Matrix form:

$$\begin{bmatrix} \$3 & \$4 & \$2 \end{bmatrix} \times \begin{bmatrix} 13 & 9 & 7 & 15 \\ 8 & 7 & 4 & 6 \\ 6 & 4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \$83 & \$63 & \$37 & \$75 \end{bmatrix}$$

$\$3 \times 13 + \$4 \times 8 + \$2 \times 6$

They sold **$83** worth of pies on Monday, **$63** on Tuesday, etc.

9 Dividing

And what about division? Well we **don't** actually divide matrices, we do it this way:

$$A \div B$$
$$= A \times \frac{1}{B}$$
$$= A \times B^{-1}$$

where $\boldsymbol{B}^{-1}$ means the "inverse" of B.

We'll learn more about the Inverse of a Matrix later.

10 Identity Matrix

The "Identity Matrix" is the matrix equivalent to the number "1":

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- It is "square" (has same number of rows and columns),
- It has **1**'s on the diagonal and **0**s everywhere else,
- It's symbol is the capital letter **I**.

It is a **special matrix** , because when you multiply by it, the original is unchanged:

A × I = A

I × A = A