Proofs

Proof by Contradiction

Suppose we have two statements A and B, and we want to prove that if A is true then B is true (we write this as $A \Rightarrow B$, and say "if A then B").

One way of doing this is to assume that A is true but that B is false, and get a contradiction.

This is the method of **proof by contradiction**.

Example 1

Consider these two statements;

A n^2 is even
B   n is even $n \in \mathbb{Z}$

Prove that $A \Rightarrow B$ by contradiction.

Proof: Assume that A is true but B is false.

A n^2 is even
B   n is odd

Then n is odd and n = 2m + 1 where $m \in \mathbb{Z}$

Then n is odd and n = 2m + 1 where $m \in \mathbb{Z}$

But: $(2m + 1)^2 = 4m^2 + 4m + 1$
$= 2(2m^2 + m) + 1$
$= 2r + 1 \quad r \in \mathbb{Z}$

This shows that $(2m + 1)^2 = n^2$ is odd, which contradicts statement A.

Hence n must be even.

Example 2

A common question in assessments is to prove that, for some non-square integer a, $\sqrt{a}$ is irrational.

The method below holds for all such a.

Example 2

Prove by contradiction that $\sqrt{7}$ is irrational.

Proof: Assume that it is rational.

Then $\sqrt{7} = \frac{m}{n}$ (m and n have no common factors)

$7 = \frac{m^2}{n^2}$

$7n^2 = m^2$ m^2 is a multiple of 7 $\Rightarrow$ m is divisible by 7

$\therefore 7 \mid m$

$\therefore 7^2 \mid m^2$

$\therefore 7^2 \mid 7n^2$

$\therefore 7 \mid n$ So, m and n have a common factor hence, by contradiction, $\sqrt{7}$ must be irrational.

Example 3

Prove by contradiction that $\sqrt{6}$ is irrational.

Proof: Assume that it is rational.

Then $\sqrt{6} = \frac{m}{n}$ (m and n have no common factors)

$6 = \frac{m^2}{n^2}$

$6n^2 = m^2$ m^2 is a multiple of 6 $\Rightarrow$ m is divisible by 6

$\therefore 6 \mid m$

$\therefore 6^2 \mid m^2$

$\therefore 6^2 \mid 6n^2$ i.e. both divisible by 6

$\therefore 6 \mid n$ So, m and n have a common factor hence, by contradiction, $\sqrt{6}$ must be irrational.

2010 Q12

Prove by contradiction that if x is an irrational number, then $2 + x$ is irrational. 4

Prove by contradiction that if x is an irrational number, then $2 + x$ is rational.

Assume $2 + x$ is rational

and let $2 + x = \frac{p}{q}$ where p, q are integers

So $x = \frac{p}{q} - 2$

$= \frac{p - 2q}{q}$ ***express as a single fraction**

Since p - 2q and q are integers, it follows that x is rational*.

This is a contradiction.

Exercise Q1

Prove that if $a \in \mathbb{Q}$ and x is irrational, then $a + x$ is irrational.

A: $a \in \mathbb{Q}$ and x is irrational; B: $a + x$ is irrational. Assume the negation of B, i.e., that $a + x$ is rational. Then $\exists s, t \in \mathbb{Z}$ (with $t \neq 0$) s.t.

$$a + x = \frac{s}{t}$$

$$x = \frac{s}{t} - a$$

$$x = \frac{s - at}{t}$$

Hence, $a \in \mathbb{Q}$ and $s, t \in \mathbb{Z}$ imply that $\frac{s - at}{t} = x \in \mathbb{Q}$, which contradicts the assumed irrationality of x.

Exercise Q2

Prove that if $x^3 + 7x > 0$, then $x > 0$.

Assume that $x^3 + 7x > 0$ and $x \leq 0$. Then, $x^3 \leq 0$ and $7x \leq 0$. Thus,

$$x^3 + 7x \leq 0 + 0$$

$$x^3 + 7x \leq 0$$

This clearly violates the assumption that $x^3 + 7x > 0$. Hence, $x > 0$.