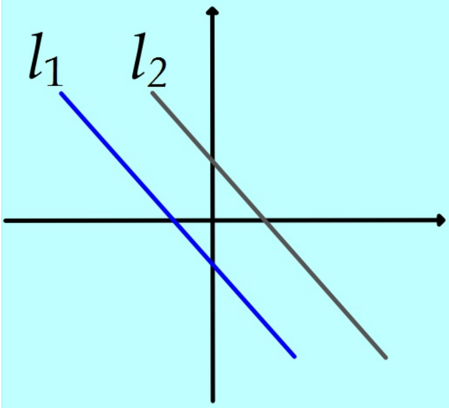
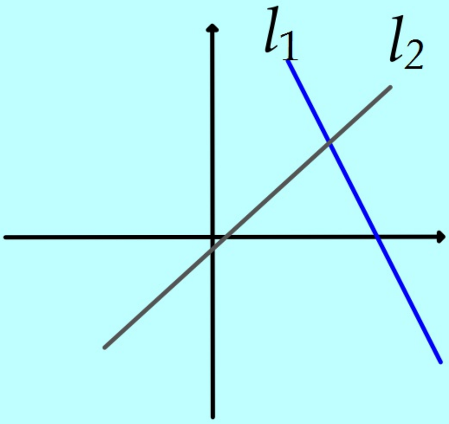
# Gaussian Elimination

# Linear Systems

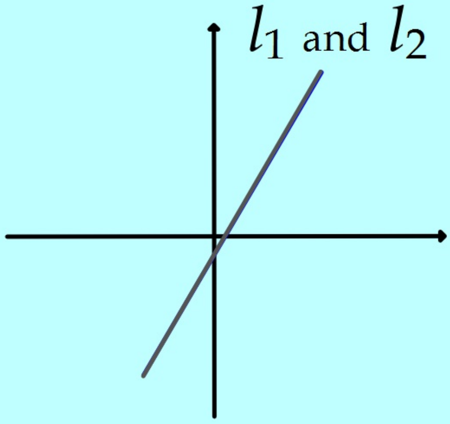
We have considered only two equations with two unknowns in the past, and will now show that the same three possibilities hold for arbitrary linear systems:




*(a)* No solution




*(b)* One solution




*(c)* Infinitely many solutions

*Every system of linear equations has no solutions, or has exactly one solution, or has infinitely many solutions.*

# Gaussian Elimination

First we need a strategy for solving systems of linear equations - Gaussian Elimination.

This is a method of using matrices to solve equations.

**Note/**

Sometimes known as Elementary Row Operations (EROs)

**Example**

Solve

$$\begin{aligned} x + 3y + 2z &= 8 \\ x + 4y + z &= 13 \\ 2x + 9y - 3z &= 35 \end{aligned}$$

The coefficients of the variables can be written in matrix form as follows.

$$\begin{pmatrix} 1 & 3 & 2 \\ 1 & 4 & 1 \\ 2 & 9 & -3 \end{pmatrix}$$

The solutions can also be written in matrix form as $\begin{pmatrix} 8 \\ 13 \\ 35 \end{pmatrix}$

These two matrices can be combined to form an **augmented matrix**.

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 8 \\ 1 & 4 & 1 & 13 \\ 2 & 9 & -3 & 35 \end{array}\right)$$

We must now use **row operations** to reduce this augmented matrix to **upper triangular form**.

## Row operations

The three ways that a matrix can be manipulated to solve a system of equations are called elementary row operations.

- Interchange two rows
- Multiply one row by a non-zero constant
- Change one row by adding or subtracting a multiple of another row.

## Upper triangular form

A matrix is in upper triangular form if all the entries below the main diagonal are zero. This can only be produced if the matrix is square.

For example, $\begin{pmatrix} 4 & 6 & 9 \\ 0 & 3 & -2 \\ 0 & 0 & -3 \end{pmatrix}$ is in upper triangular form.

Back to Ex 1

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 8 \\ 1 & 4 & 1 & 13 \\ 2 & 9 & -3 & 35 \end{array}\right) \quad \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$$R_3 \rightarrow R_3 - 2R_1 \quad \left(\begin{array}{ccc|c} 1 & 3 & 2 & 8 \\ 1 & 4 & 1 & 13 \\ 0 & 3 & -7 & 19 \end{array}\right)$$

$$R_2 \rightarrow R_2 - R_1 \quad \left(\begin{array}{ccc|c} 1 & 3 & 2 & 8 \\ 0 & 1 & -1 & 5 \\ 0 & 3 & -7 & 19 \end{array}\right)$$

$$R_3 \to R_3 - 3R_2 \quad \left(\begin{array}{ccc|c} 1 & 3 & 2 & 8 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & -4 & 4 \end{array}\right)$$

The solutions can now be found using back substitution

From $R_3$

$$-4z = 4$$
$$z = -1$$

From $R_2$

$$y - z = 5$$
$$y + 1 = 5$$
$$y = 4$$

From $R_1$

$$x + 3y + 2z = 8$$
$$x + 12 - 2 = 8$$
$$x = -2$$

$$\boxed{x = -2 \quad y = 4 \quad z = -1}$$

1. $$x + 2y + 3z = 12$$
$$x + 3y + z = 3$$
$$2x + 5y + 2z = 7$$

$x$ = 2, $y$ = -1 and $z$ = 4

2. $$x + 2y - z = 3$$
$$2x + 5y + 2z = -3$$
$$4x - 2y + z = 12$$

$x$ = 3, $y$ = -1 and $z$ = -2

3. $$x + y - 2z = 5$$
$$2x + 4y + z = 15$$
$$-3x + 2y + 2z = 14$$

$x$ = -2, $y$ = 5 and $z$ = -1

4. $$2x + y + 4z = -1$$
$$x + 2y - 3z = 8$$
$$3x - 2y + 2z = 10$$

$x$ = 4, $y$ = -1 and $z$ = -2

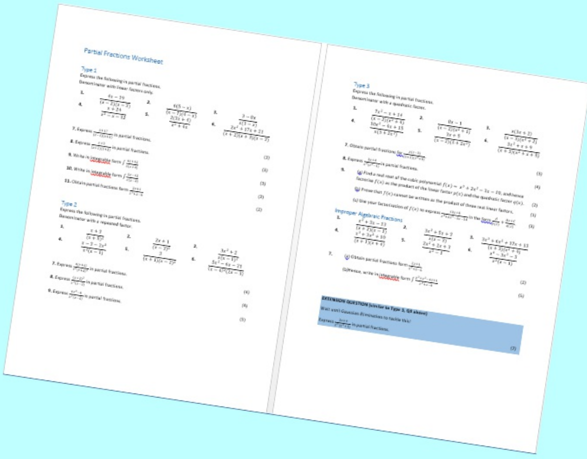
5. $$x + 2y = 10$$
$$2x - 3y + z = 3$$
$$x - 4z = 18$$

$x$ = 2, $y$ = -1 and $z$ = 4

**EXTENSION QUESTION (similar to Type 3, Q9 above)**

Wait until Gaussian Elimination to tackle this!

Express $\dfrac{3x+4}{x^2\left(x^2+1\right)}$ in partial fractions. (7)



Partial Fractions Worksheet

Type 1

Type 2

Type 3

Improper Algebraic Fractions