Integration Techniques

Recall the Standard Integrals:

$$\int \frac{1}{x}\,dx = \ln|x| + C$$
$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

- Using Division to help Integrate   Remember $Q + \frac{R}{D}$
- Using Partial Fractions to Integrate
- A Combination of Division and Partial Fractions

Ex 1 $\int \frac{x^2 - x + 3}{x-1}\,dx$

$$\int \frac{x^2 - x + 3}{x-1}\,dx = \int x + \frac{3}{x-1}\,dx$$
$$= \frac{1}{2}x^2 + 3\ln|x-1| + C$$

Ex 2 (a) Express $\frac{1}{x^2 - x - 6}$ in partial fractions.

(b) Hence find the exact value of $\int_0^1 \frac{1}{x^2 - x - 6}\,dx$ giving your answer in the form $k\ln a$.

(a) $$\frac{1}{x^2 - x - 6} = \frac{\frac{1}{5}}{x-3} - \frac{\frac{1}{5}}{x+2} = \frac{1}{5(x-3)} - \frac{1}{5(x+2)}$$

(b) $$\int_0^1 \frac{1}{x^2 - x - 6}\,dx = \int_0^1 \left(\frac{\frac{1}{5}}{x-3} - \frac{\frac{1}{5}}{x+2} \right) dx$$

$$= \left[\frac{1}{5}\ln|x-3| - \frac{1}{5}\ln|x+2| \right]_0^1$$

$$= \frac{1}{5}\Big[\big[\ln|-2| - \ln|3|\big] - \big[\ln|-3| - \ln|2|\big] \Big]$$

$$= \frac{1}{5}\Big[\big[\ln|2| - \ln|3|\big] - \big[\ln|3| - \ln|2|\big] \Big]$$

$$= \frac{1}{5}\left[2\ln|2| - 2\ln|3|\right]$$

$$= \frac{2}{5}\left[\ln|2| - \ln|3|\right]$$

$$= \frac{2}{5}\ln\left(\frac{2}{3}\right)$$

Ex 3 (a) Express $\frac{x}{x^2-1}$ in partial fractions.

(b) Hence find $\int \frac{x^3}{x^2-1}dx$

(a) $$\frac{x}{x^2-1} = \frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x+1} = \frac{1}{2(x-1)} + \frac{1}{2(x+1)}$$

(b) Note that $\frac{x^3}{x^2-1}$ is an improper rational function and algebraic long division must be used before integration.

$$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$$

$$= x + \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$$ (the partial fractions from (a))

Hence $$\int \frac{x^3}{x^2-1}dx = \int \left(x + \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1} \right) dx$$

$$= \frac{1}{2}x^2 + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

AH Maths 2007

4. Express $\dfrac{2x^2-9x-6}{x(x^2-x-6)}$ in partial fractions. **3**

Given that

$$\int_4^6 \frac{2x^2-9x-6}{x(x^2-x-6)}\,dx = \ln\frac{m}{n},$$

determine values for the integers m and n. **3**

$$\frac{2x^2-9x-6}{x(x^2-x-6)} = \frac{2x^2-9x-6}{x(x+2)(x-3)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-3} \qquad \mathbf{1}$$

$$2x^2-9x-6 = A(x+2)(x-3) + Bx(x-3) + Cx(x+2)$$

$$\begin{aligned} x = 0 &\Rightarrow -6A = -6 \Rightarrow A = 1\\ x = -2 &\Rightarrow 10B = 20 \Rightarrow B = 2\\ x = 3 &\Rightarrow 15C = -15 \Rightarrow C = -1 \end{aligned} \qquad \mathbf{2E1}$$

$$\therefore \frac{2x^2-9x-6}{x(x^2-x-6)} = \frac{1}{x} + \frac{2}{x+2} - \frac{1}{x-3}$$

$$\begin{aligned} \int_4^6 \frac{2x^2-9x-6}{x(x^2-x-6)} &= \int_4^6 \left(\frac{1}{x} + \frac{2}{x+2} - \frac{1}{x-3}\right) dx\\ &= \left[\ln x + 2\ln(x+2) - \ln(x-3)\right]_4^6 \qquad \mathbf{2E1}\\ &= \left[\ln\frac{x(x+2)^2}{(x-3)}\right]_4^6\\ &= \ln\frac{6\times 64}{3} - \ln\frac{4\times 36}{1}\\ &= \ln\frac{2\times 64}{4\times 36} = \ln\frac{8}{9} \qquad \mathbf{1} \end{aligned}$$

AH Maths 2008

4. Express $\dfrac{12x^2+20}{x(x^2+5)}$ in partial fractions. **3**

Hence evaluate

$$\int_1^2 \frac{12x^2+20}{x(x^2+5)}\,dx. \qquad \mathbf{3}$$

$$\frac{12x^2+20}{x(x^2+5)} = \frac{A}{x} + \frac{Bx+C}{x^2+5} \qquad \mathbf{1M}$$

$$\begin{aligned} 12x^2+20 &= A(x^2+5) + x(Bx+C)\\ &= (A+B)x^2 + Cx + 5A \end{aligned}$$

$$\therefore \quad 5A = 20 \Rightarrow A = 4 \Rightarrow B = 8$$

$$C = 0 \qquad \mathbf{2E1}$$

Hence $$\frac{12x^2+20}{x(x^2+5)} = \frac{4}{x} + \frac{8x}{x^2+5}$$

$$\begin{aligned} \int_1^2 \frac{12x^2+20}{x(x^2+5)}\,dx &= \int_1^2 \frac{4}{x} + \frac{8x}{x^2+5}\,dx\\ &= \int_1^2 \frac{4}{x}\,dx + 4\int_1^2 \frac{2x}{x^2+5}\,dx\\ &= \left[4\ln x + 4\ln(x^2+5)\right]_1^2 \qquad \mathbf{1,1}\\ &= 4\left[\ln x(x^2+5)\right]_1^2 = 4\left[\ln 18 - \ln 6\right] \qquad \mathbf{1}\\ &= 4\ln 3\ (\approx 4{\cdot}39) \end{aligned}$$

AH Maths 2015

17. Find $\int \frac{2x^3 - x - 1}{(x-3)(x^2+1)}\,dx,\ x > 3.$ **9**

$x^3 - 3x^2 + x - 3 \overline{\big)\,2x^3 + 0x^2 - x - 1}$ with quotient 2 $2x^3 - 6x^2 + 2x - 6$ $6x^2 - 3x + 5$	9	•[1] for knowing to divide and starting division •[2] correct division[1].
$\int \frac{2x^3 - x - 1}{(x-3)(x^2+1)}dx = \int \left(2 + \frac{6x^2 - 3x + 5}{(x-3)(x^2+1)}\right)dx$ $= \int 2 + \frac{A}{x-3} + \frac{Bx+C}{x^2+1}dx$		•[3] for correct form of PFs[5].
$6x^2 - 3x + 5 = A(x^2+1) + (Bx+C)(x-3)$		•[4] creating correct equation
$x = 0 \quad 5 = A - 3C$ $x = 3 \quad 50 = 10A \quad \Rightarrow A = 5$ $C = 0$		•[5] for any two values[4].
$x = 1 \quad 8 = 2A - 2B - 2C$ $8 = 10 - 2B \quad \Rightarrow B = 1$		•[6] for third value[4].
$\int \frac{2x^3 - x - 1}{(x-3)(x^2+1)}dx = \int 2 + \frac{5}{x-3} + \frac{x}{x^2+1}dx$ $= 2x + 5\ln\lvert x-3 \rvert + \frac{1}{2}\ln(x^2+1) + k$		•[7] for putting into integral *and* any one term correctly integrated[3]. •[8] for any second term. •[9] for third term and $+k$.[2]