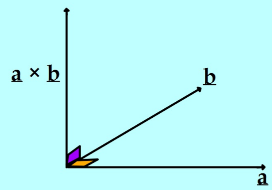
Vectors

The Vector Product

a × **b** (**a** cross **b**) is defined to be the vector **(not scalar)**



(1) with direction perpendicular to both **a** and **b**

(2) with magnitude $|\underline{a} \times \underline{b}| = |\underline{a}|\,|\underline{b}|\sin\theta$
*(where θ is the angle between **a** and **b**)*

Note/ A vector product can be calculated using the same method as that for the determinant of a 3 × 3 matrix.

Component Form of Vector Product

If $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

then $\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Example

$\underline{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

(i) $\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix}$

$= \underline{i}\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} - \underline{j}\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} + \underline{k}\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$

$= (2 + 3)\underline{i} - (1 - 6)\underline{j} + (-1 - 4)\underline{k}$

$= 5\underline{i} + 5\underline{j} + -5\underline{k}$ or $\begin{pmatrix} 5 \\ 5 \\ -5 \end{pmatrix}$

Example

$\underset{\sim}{a} = \begin{pmatrix}1\\2\\3\end{pmatrix} \qquad \underset{\sim}{b} = \begin{pmatrix}2\\-1\\1\end{pmatrix}$

(ii) $\underset{\sim}{b} \times \underset{\sim}{a} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

$= \underset{\sim}{i}\begin{vmatrix}-1 & 1\\ 2 & 3\end{vmatrix} - \underset{\sim}{j}\begin{vmatrix}2 & 1\\ 1 & 3\end{vmatrix} + \underset{\sim}{k}\begin{vmatrix}2 & -1\\ 1 & 2\end{vmatrix}$

$= (-3 - 2)\underset{\sim}{i} - (6 - 1)\underset{\sim}{j} + (4 - (-1))\underset{\sim}{k}$

$= -5\underset{\sim}{i} - 5\underset{\sim}{j} + 5\underset{\sim}{k}$ or $\begin{pmatrix}-5\\-5\\5\end{pmatrix}$

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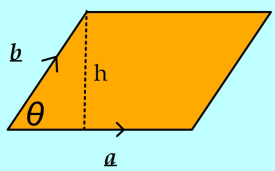
Note/

1. $\underset{\sim}{a} \times \underset{\sim}{b} = \underset{\sim}{0} \rightarrow$ **a** is parallel to **b** (or, either **a** or **b** = **0**)

$$\underset{\sim}{a} \times \underset{\sim}{b} = 0$$
$$|a||b|\sin\theta = 0$$
$$\sin\theta = 0$$
$$\theta = 0$$

Note/

2. $|\underset{\sim}{a} \times \underset{\sim}{b}|$ is the area of a parallelogram with sides $\underset{\sim}{a}$ and $\underset{\sim}{b}$



Area = base × height

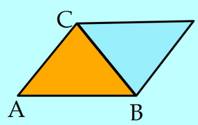
$= |a||b|\sin\theta$

$\sin\theta = \dfrac{h}{|b|}$

$h = |\underset{\sim}{b}|\sin\theta$

Example

Find the area of the triangle with vertices A(1, 3, -2), B(4, 3, 0) and C(2, 1, 1).



Area △ $= \dfrac{1}{2}$ Area ▱

$= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a} = \begin{pmatrix}3\\0\\2\end{pmatrix} \qquad \overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a} = \begin{pmatrix}1\\-2\\3\end{pmatrix}$

$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ $\overrightarrow{AC} = \underline{c} - \underline{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 0 & 2 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= \underline{i}\begin{vmatrix} 0 & 2 \\ -2 & 3 \end{vmatrix} - \underline{j}\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} + \underline{k}\begin{vmatrix} 3 & 0 \\ 1 & -2 \end{vmatrix}$$

$$= -4\underline{i} - 7\underline{j} - 6\underline{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = -4\underline{i} - 7\underline{j} - 6\underline{k}$$

$$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-4)^2 + (-7)^2 + (-6)^2}$$

$$= \sqrt{16 + 49 + 36}$$

$$\text{Area}_{\triangle} = \tfrac{1}{2}\sqrt{101} \text{ units}^2$$

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Example

Find the perpendicular unit vector to both

a = 2***i*** + ***j*** - ***k***
b = ***i*** - ***j*** + 2***k***.

a × ***b*** is perp. to both ***a*** and ***b***.

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= (2-1)\underline{i} - (4-(-1))\underline{j} + (-2-1)\underline{k}$$

$$= \underline{i} - 5\underline{j} - 3\underline{k}$$

$$= (2-1)\underline{i} - (4-(-1))\underline{j} + (-2-1)\underline{k}$$

$$= \underline{i} - 5\underline{j} - 3\underline{k}$$

Need a unit vector:

$$|\underline{a} \times \underline{b}| = \sqrt{1^2 + (-5)^2 + (-3)^2}$$

$$= \sqrt{35}$$

$$\underline{U}_{\underline{a} \times \underline{b}} = \frac{1}{\sqrt{35}}(\underline{i} - 5\underline{j} - 3\underline{k})$$