Differentiation

1 Differentiation of Trigonometric Functions

As well as $\sin x$ and $\cos x$, there are some other trigonometric functions.

$$\tan x = \frac{\sin x}{\cos x}$$

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

Example

$y = tanx = \frac{sinx}{cosx}$ where $u = sinx$ $u' = cosx$

$v = cosx$ $v' = -sinx$

$$\frac{dy}{dx} = \frac{u'v - v'u}{v^2}$$

$$= \frac{cosx \bullet cosx - (-sinx) \bullet sinx}{cos^2 x}$$

$$= \frac{cos^2 x + sin^2 x}{cos^2 x}$$

$$= \frac{1}{cos^2 x}$$

$$= \underline{\underline{sec^2 x}}$$

The other trigonometric functions can be differentiated in a similar way.

Summary

$f(x)$	$f'(x)$
$sinx$	$cosx$
$cosx$	$-sinx$
$tanx$	$sec^2 x$
$cosecx$	$-cosecxcotx$
$secx$	$secxtanx$
$cotx$	$-cosec^2 x$

Examples

(a) $cosec(\frac{\pi}{3})$

$= \frac{1}{sin(\frac{\pi}{3})}$

$= \frac{2}{\sqrt{3}}$

(b) $cot(\frac{\pi}{4})$

$= \frac{1}{tan(\frac{\pi}{4})}$

$= 1$

(c) $f(x) = 3\tan(4x)$

$f^{'}(x) = 12sec^2(4x)$

(d) $f(x) = -2\text{cosec}x$

$f^{'}(x) = 2cosecxcotx$

(e) $f(x) = -\frac{1}{2}\sec(3x)$

$f^{'}(x) = -\frac{3}{2}\sec(3x)\tan(3x)$

(f) $y = x^3\sec\left(x^2\right)$ where $u = x^3$ $u' = 3x^2$

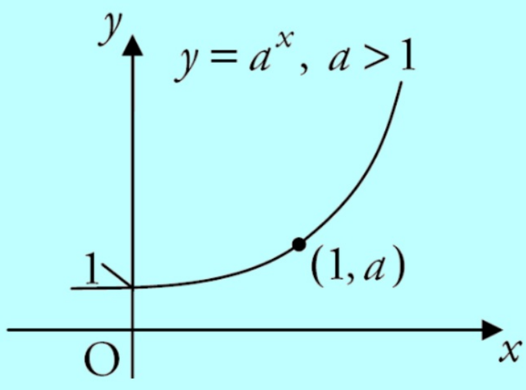
$v = sec\left(x^2\right)$ $v' = sec\left(x^2\right)tan\left(x^2\right)$

$$\frac{dy}{dx} = u'v + v'u$$

$$= 3x^2 \bullet sec\left(x^2\right) + 2xsec\left(x^2\right)tan\left(x^2\right) \bullet x^3$$

$$= \underline{\underline{x^2sec\left(x^2\right)\left[3 + 2x^2tan\left(x^2\right)\right]}}$$

2 Differentiation of Exponential Functions



Note; Differentiating e^x does not change the functions e^x is the ONLY function with this property.

Example 1

$$y = e^{2x}$$

$$\frac{dy}{dx} = \underline{\underline{2e^{2x}}}$$

Example 2

$$y = e^{5-3x}$$

$$\frac{dy}{dx} = \underline{\underline{-3e^{5-3x}}}$$

Example 3

$$y = e^{x^2} + \frac{4}{e^{2x}}$$

$$= e^{x^2} + 4e^{-2x}$$

$$\frac{dy}{dx} = e^{x^2} \bullet 2x + 4e^{-2x} \bullet (-2)$$

$$= 2xe^{x^2} - 8e^{-2x}$$

$$= \underline{\underline{2xe^{x^2} - \frac{8}{e^{2x}}}}$$

Example 4

$y = x^3 e^{4x}$ where $u = x^3$ $\quad u' = 3x^2$

$v = e^{4x}$ $\quad v' = 4e^{4x}$

$$\frac{dy}{dx} = u'v + v'u$$

$$= 3x^2 \bullet e^{4x} + 4e^{4x} \bullet x^3$$

$$= \underline{\underline{x^2 e^{4x}(3 + 4x)}}$$

Example 5

$y = \frac{x^3}{e^{2x}}$ where $u = x^3$ $\quad u' = 3x^2$

$v = e^{2x}$ $\quad v' = 2e^{2x}$

$$\frac{dy}{dx} = \frac{u'v - v'u}{v^2}$$

$$= \frac{3x^2 \bullet e^{2x} - 2e^{2x} \bullet x^3}{\left(e^{2x}\right)^2}$$

$$= \underline{\underline{\frac{x^2(3 - 2x)}{e^{2x}}}}$$

Example 6

If $y = e^{-x}\cos x$, find the second derivative $\frac{d^2y}{dx^2}$

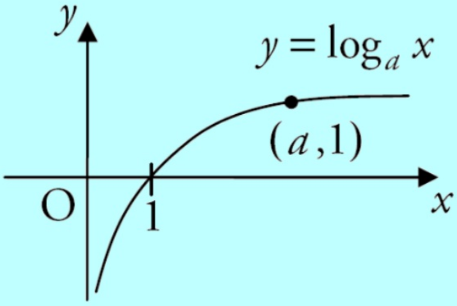
$$y = e^{-x}\cos x \quad \text{where} \quad u = e^{-x} \quad u' = -e^{-x}$$
$$v = \cos x \quad v' = -\sin x$$

$$\frac{dy}{dx} = u'v + v'u$$
$$= -e^{-x} \bullet \cos x - \sin x \bullet e^{-x}$$
$$= -e^{-x}(\cos x + \sin x)$$

$$\frac{dy}{dx} = -e^{-x}(\cos x + \sin x) \quad \text{where} \quad u = -e^{-x} \quad u' = e^{-x}$$
$$v = \cos x + \sin x \quad v' = -\sin x + \cos x$$

$$\frac{d^2y}{dx^2} = u'v + v'u$$
$$= e^{-x} \bullet (\cos x + \sin x) + (\cos x - \sin x) \bullet e^{-x}$$
$$= \underline{\underline{2e^{-x}\sin x}}$$

3 Differentiation of Logarithmic Functions



Note: We can only find lnx of positive values of x, strictly speaking we should only write $y = |ln|x$ or $y = |log_e|x$.

Recall the following facts about natural logarithms:

(1) $lnx = log_e x$ where $e = 2.71828$

(2) lnx and e^x are inverse functions. This means that;

(3) $e^{lnx} = x \quad x > 0$

(4) $ln(e^x) = x$

The derivative of lnx (x > 0) can be found as follows.

Let $y = lnx$

$log_e x = y$

$x = e^y$

Differentiate both sides with respect to y

$\Rightarrow \frac{dx}{dy} = e^y$

So, $= \frac{1}{e^y} = \frac{1}{x}$

Hence $\frac{d}{dx} lnx = \frac{1}{x} \quad x > 0$

Example 1

$y = \ln(3x - 1)$

$$\frac{dy}{dx} = \frac{1}{3x-1} \times 3x$$

$$= \frac{3x}{3x-1}$$

Example 2

$y = \ln(4x^2 + 1)$

$$\frac{dy}{dx} = \frac{1}{4x^2+1} \times 8x$$

$$= \frac{8x}{4x^2+1}$$

Example 3

$y = \ln(cos4x)$

$$\frac{dy}{dx} = \frac{1}{cos4x} \times (-4sin4x)$$

$$= \frac{-4sin4x}{cos4x}$$

$$= -4tan4x$$

Example 4

$$f(x) = \ln(e^{2x} + 1)$$

$$f'(x) = \frac{1}{e^{2x}+1} \times 2e^{2x}$$

$$= \underline{\underline{\frac{2e^{2x}}{e^{2x}+1}}}$$

Example 5

$y = x^3 lnx$ where $u = x^3$ $u' = 3x^2$

$v = lnx$ $v' = \frac{1}{x}$

$$\frac{dy}{dx} = u'v + v'u$$

$$= 3x^2 \bullet lnx + \frac{1}{x} \bullet x^3$$

$$= \underline{\underline{x^2(3lnx + 1)}}$$

Example 6

$$y = \frac{lnx}{e^{2x}} \quad \text{where} \quad u = lnx \quad u' = \frac{1}{x}$$

$$v = e^{2x} \quad v' = 2e^{2x}$$

$$\frac{dy}{dx} = \frac{u'v - v'u}{v^2}$$

$$= \frac{\frac{1}{x} \bullet e^{2x} - 2e^{2x} \bullet lnx}{\left(e^{2x}\right)^2}$$

$$= \frac{1 - 2xlnx}{xe^{2x}}$$

The laws of logarithms can sometimes be used to simplify functions before differentiation. Recall the three laws of logs below:

$$\ln(ab) = lna + lnb$$

$$\ln\left(\frac{a}{b}\right) = lna - lnb$$

$$lna^m = mlna$$

Example 1

Given $y = \ln\left(x^4 e^{2x}\right)$, find $\frac{dy}{dx}$.

$$y = lnx^4 + lne^{2x}$$

$$= 4lnx + 2x$$

$$\frac{dy}{dx} = 4 \bullet \frac{1}{x} + 2$$

$$= \frac{4}{x} + 2$$

Example 2

Given $y = ln\left(\frac{x}{2x+1}\right)$, $x > 0$ show that $\frac{dy}{dx} = \frac{1}{x(2x+1)}$

$$y = ln\left(\frac{x}{2x+1}\right)$$

$$y = lnx - \ln(2x + 1)$$

$$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{2x+1} \bullet 2$$

$$= \frac{1}{x} - \frac{2}{2x+1}$$

$$= \frac{1}{x(2x+1)}$$ as required.

Example 3

Given $y = ln\sqrt{2x^3 + 1}$, $x > 0$ find $\frac{dy}{dx}$.

$$y = ln\left(2x^3 + 1\right)^{\frac{1}{2}}$$

$$= \tfrac{1}{2}\ln(2x^3 + 1)$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{2x^3+1} \times 6x^2$$

$$= \underline{\underline{\frac{3x^2}{2x^3+1}}}$$