Algebraic Skills

Factorials and Binomial Coefficient

Factorials

$n!$ (known as n factorial) is the product of the integers $n, n-1, n-2, \ldots, 2, 1$.

$n! = n \times (n-1) \times (n-2) \times (n-3) \ldots \times 3 \times 2 \times 1$

Zero factorial $0! = \underline{\underline{1}}$

Ex1.
Calculate the value of 5!

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = \underline{\underline{120}}$$

Note

$(n-1)!$ is the product of the integers $(n-1), (n-2), (n-3), \ldots, 2, 1$.

So $\dfrac{n!}{(n-1)!} = \dfrac{n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1}{(n-1) \times (n-2) \times \cdots \times 2 \times 1} = \underline{\underline{n}}$

This gives $n! = n \times (n-1)!$

Ex 2.
What is 10! in terms of 9!

$$10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \underline{\underline{10 \times 9!}}$$

Binomial Coefficient

The number given by $\dfrac{n!}{r!(n-r)!}$ is called a **Binomial Coefficient.**

It is denoted by $\binom{n}{r}$ or ${}^nC_r$.

$\binom{n}{r}$ or ${}^nC_r$ can be defined as the number of ways selecting r values from n values (the different number of combinations)

Ex 3.
In the national lottery there are 49 numbers from which to pick a set of 6 numbers.
How many different sets of 6 numbers can be picked?

$$\binom{49}{6} \text{ or } {}^{49}C_6 = \frac{n!}{r!(n-r)!} = \frac{49!}{6!(49-6)!} = \frac{49!}{6!43!} = \underline{\underline{13\ 983\ 816}}$$

Ex 4.

What is the value of $\binom{6}{3}$?

$$\binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{6!}{3!\times 3!} = \frac{6\times5\times4\times3\times2\times1}{3\times2\times1\times3\times2\times1} = \frac{6\times5\times4}{3\times2\times1} = \frac{120}{6} = \underline{\underline{20}}$$

Rule 1 for binomial coefficients.

$$\binom{n}{r} = \binom{n}{n-r}$$

Ex 5.

Find another binomial coefficient equal to $\binom{8}{3}$.

$$\binom{n}{r} = \binom{n}{n-r} \Rightarrow \binom{8}{3} = \binom{8}{8-3}$$

$$= \underline{\underline{\binom{8}{5}}}$$

Rule 2 for binomial coefficients.

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

Ex 6.

Write down $\binom{9}{2} + \binom{9}{3}$ as a binomial coefficient.

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r} \Rightarrow \binom{9}{2} + \binom{9}{3} = \underline{\underline{\binom{10}{3}}}$$

It is also possible to find the value of n if you are given the value of a binomial coefficient and the value of r.

Ex 7.

Find a positive integer n such that $\binom{n}{2} = 6$

$$6 = \binom{n}{2} = \frac{n!}{2!(n-2)!}$$

$$6 = \frac{n\times(n-1)}{2}$$

$$12 = n(n-1)$$

$$n^2 - n - 12 = 0$$

$$(n-4)(n+3) = 0$$

The only solution which is a positive integer is $\underline{\underline{n = 4}}$

Factorials and Binomial Coefficients Worksheet

2010 Q5 4 marks

Show that

$$\binom{n+1}{3} - \binom{n}{3} = \binom{n}{2}$$

where the integer n is greater than or equal to 3.

$\binom{n+1}{3} - \binom{n}{3} = \frac{(n+1)!}{3!(n-2)!} - \frac{n!}{3!(n-3)!}$	**1**	both terms correct
$= \frac{(n+1)!}{3!(n-2)!} - \frac{n!(n-2)}{3!(n-2)!}$		{alternative methods will appear}
$= \frac{(n+1)! - n!(n-2)}{3!(n-2)!}$		
$= \frac{n![(n+1)-(n-2)]}{3!(n-2)!}$	**1** **1**	correct numerator correct denominator
$= \frac{n!\times 3}{3!(n-2)!} = \frac{n!}{2!(n-2)!}$	**1**	
$= \binom{n}{2}$		**1** for knowing (anywhere) $(n-2)! = (n-2)\times(n-3)!$