Proofs

Principle of Mathematical Induction (PMI)

Suppose we know two facts:

(1) P(1) is true;
(2) whenever P(k) is true, P(k + 1) is also true.

Then P(n) is true for all positive integers n.

In applying this Principle, we often refer to:
(1) as the base step,
(2) as the inductive step.

Note/ the assumption that P(k) is true is called the inductive hypothesis.

Induction is a method of proving an implication by the following:

1) Prove the implication is true for some starting value (usually 1)

2) Assume the implication is true for some succeeding value (usually *k*)

3) Prove that if the implication is true for the value *k*, it is also true for the value *k* + 1.

Note/ the combination of (1) and (3) above proves the implication for all values.

$\forall x$ *"for all x"*

Example 1

Prove by induction that $n^3 + 2n$ is divisible by 3 $\forall n \geq 1,\ n \in N$

Note/ the above can also be written as:
(1) $3 \mid n^3 + 2n$
(2) 3 is a factor of $n^3 + 2n$

Prove true for n = 1

$1^3 + 2(1) = 3$ which is divisible by 3.
$\therefore$ true for n = 1

Assume true for $n = k \quad \Rightarrow \quad k^3 + 2k = 3t \qquad t \in N$

Assume true for n = k $\Rightarrow$ $k^3 + 2k = 3t \quad t \in N$

Prove true for n = $k + 1$

$$(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$$
$$= k^3 + 2k + 3k^2 + 3k + 3$$
$$= 3t + 3(k^2 + k + 1)$$
$$= 3(t + k^2 + k + 1) \quad \text{which is divisible by 3.}$$

Hence if true for n = k, it is also true for n = $k + 1$.

Since true for n = 1 then by PMI $n^3 + 2n$ is divisible by 3 $\forall n \geq 1,\ n \in N$

Example 2

Prove by induction that $2^n > n \ \forall n \in N$

Prove true for n = 1

LHS : $2^1 = 2$ RHS : 1

LHS = RHS

$\therefore$ true for n = 1

Assume true for n = k $\Rightarrow$ $2^k > k$

Assume true for n = k $\Rightarrow$ $2^k > k$

Prove true for n = $k + 1$

$$2^{k+1} = 2^k \times 2^1$$
$$> k \times 2 \quad \text{since } 2^k > k$$
$$> 2k$$
$$> k + 1 \quad \text{since } k \geq 1$$

Hence if true for n = k, it is also true for n = $k + 1$.

Since true for n = 1 then by PMI $2^n > n \ \forall n \in N$

Example 3

Prove by induction that $8 \mid 3^{2n} - 1 \ \forall n \in N$

Prove true for n = 1

$3^{2(1)} - 1 = 3^2 - 1 = 8$ which is divisible by 8

$\therefore$ true for n = 1

Assume true for n = k $\Rightarrow$ $3^{2k} - 1 = 8t$

Assume true for n = k $\Rightarrow$ $3^{2k} - 1 = 8t$

Prove true for n = $k + 1$

$3^{2(k+1)} - 1 = 3^{2k+2} - 1$

$= 3^{2k} \times 3^{2} - 1$
$= 9 \times 8t$

$= 72t$ which is divisible by 8.

Hence if true for n = k, it is also true for n = $k + 1$.

Since true for n = 1 then by PMI $8 | 3^{2n} - 1 \quad \forall n \in N$