Integration by Parts

The idea of Integration by Part is to produce an integral which is easier to deal with. We use it for 'Products'.

The mnemonic LIPET can be useful in making your choice of function for differentiating.

Log
Inverse Trig
Polynomials
Exponential
Trig

Proof

Let u and v be functions of x.
The product rule (for differentiation)

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrate both sides with respect to x.

$$uv = \int u\frac{dv}{dx} + \int v\frac{du}{dx}$$

$$\Rightarrow \quad \int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$$

or $\quad I = uv - \int vu'$

Ex 1 $\quad I = \int x \sin x dx$
d i

Log
Inverse Trig
Polynomials
Exponential
Trig

$u = x \qquad v' = \sin x$
$u' = 1 \qquad v = -\cos x$

$$I = uv - \int vu'$$
$$= -x\cos x - \int -\cos x dx$$
$$= \underline{-x\cos x + \sin x + C}$$

Ex 2 $\quad I = \int xe^{6x} dx$
d i

Log
Inverse Trig
Polynomials
Exponential
Trig

$u = x \qquad v' = e^{6x}$
$u' = 1 \qquad v = \frac{1}{6}e^{6x}$

$$I = uv - \int vu'$$
$$= \frac{1}{6}xe^{6x} - \frac{1}{6}\int e^{6x} dx$$
$$= \underline{\frac{1}{6}xe^{6x} - \frac{1}{36}e^{6x} + C}$$

Ex 3 $I = \int (2x+1)(3x+2)^4 \, dx$

d i

Log
Inverse Trig
Polynomials
Exponential
Trig

$u = (2x+1) \qquad v' = (3x+2)^4$

$u' = 2 \qquad v = \frac{1}{15}(3x+2)^5$

$I = uv - \int vu'$

$= \frac{1}{15}(2x+1)(3x+2)^5 - \frac{2}{15}\int (3x+2)^5 dx$

$= \frac{1}{15}(2x+1)(3x+2)^5 - \frac{2}{270}(3x+2)^6 + C$

Ex 4 $I = \int x \ln x dx$

i d

We must select $\ln x$ as the function to be differentiated, as the standard integral for $\int \ln x \, dx$ is not known.

$u = \ln x \qquad v' = x$

$u' = \frac{1}{x} \qquad v = \frac{1}{2}x^2$

$I = uv - \int vu'$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int \frac{1}{x} x^2 dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$

Ex 5 $I = \int \ln x \, dx = \int \ln x . 1 \, dx$

d i

$u = \ln x \qquad v' = 1$

$u' = \frac{1}{x} \qquad v = x$

$I = uv - \int vu'$

$= x \ln x - \int 1 dx$

$= x \ln x - x + C$

Integration by Parts Worksheet Q1 – Q8

Repeated Integration by Parts

Log
Inverse Trig
Polynomials
Exponential
Trig

Ex 6 $I = \int x^2 e^x dx$

d i

$u = x^2 \qquad v' = e^x$

$u' = 2x \qquad v = e^x$

$I = uv - \int vu'$

$= x^2 e^x - \int 2xe^x dx$

$u = 2x \qquad v' = e^x$

$u' = 2 \qquad v = e^x$

$$u = 2x \qquad v' = e^x$$
$$u' = 2 \qquad v = e^x$$

$$= x^2e^x - (\,2xe^x - \int 2e^x\,dx\,)$$
$$= \underline{x^2e^x - 2xe^x + 2e^x + C}$$

Ex 7 $I = \int x^2 \sin x dx$
d i

$$u = x^2 \qquad v' = \sin x$$
$$u' = 2x \qquad v = -\cos x$$

$$I = uv - \int vu'$$
$$= -x^2 \cos x - \int -2x \cos x dx$$
$$= -x^2 \cos x + \int 2x \cos x dx$$
$$u = 2x \qquad v' = \cos x$$
$$u' = 2 \qquad v = \sin x$$
$$= -x^2 \cos x + 2x\sin x - \int 2 \sin x dx$$
$$= \underline{-x^2 \cos x + 2x \sin x + 2\cos x + C}$$

Ex 8 Evaluate the definite integral $\int_0^{\frac{\pi}{2}} x \sin 2x\, dx$
d i

$$I = \int x \sin 2x\, dx$$

$$u = x \qquad v' = \sin 2x$$
$$u' = 1 \qquad v = -\frac{1}{2}\cos 2x$$

$$I = uv - \int vu'$$

$$= -\frac{1}{2}x\cos 2x - \int -\frac{1}{2}\cos 2x\, dx$$

$$= -\frac{1}{2}x\cos 2x + \int \frac{1}{2}\cos 2x\, dx$$

$$= -\frac{1}{2}x\cos 2x + \frac{1}{2}.\frac{1}{2}\sin 2x + C$$

$$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$$

$$\int_0^{\frac{\pi}{2}} x\sin 2x\,dx = \left[-\frac{1}{2}x\cos 2x+\frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= \left[-\frac{1}{2}.\frac{\pi}{2}.\cos\pi+\frac{1}{4}\sin\pi\right]-\left[-\frac{1}{2}.0.\cos 0+\frac{1}{4}\sin 0\right]$$

$$= \left[-\frac{\pi}{4}.(-1)+\frac{1}{4}.(0)\right]-[0]$$

$$= \frac{\pi}{4}$$

Ex 9 Evaluate the definite integral $\int_0^1 xe^{-2x}dx$ leaving your answer in terms of e.

d i

$$I = \int xe^{-2x}\,dx$$

$u = x$ $\quad$ $v' = e^{-2x}$

$u' = 1$ $\quad$ $v = -\frac{1}{2}e^{-2x}$

$$I = uv - \int vu'$$

$$= -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\,dx$$

$$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$$

$$\int_0^1 xe^{-2x}dx = \left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^1$$

$$= \left[-\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2}\right] - \left[-\frac{1}{4}\right]$$

$$= -\frac{3}{4}e^{-2} + \frac{1}{4}$$

$$= \frac{1}{4}\left(1-3e^{-2}\right)$$

Integration by Parts Worksheet Q9 - Q12

Ex 10 Use integration by parts to show that

$$\int e^x \cos x \, dx = \frac{1}{2}e^x(\sin x + \cos x) + C$$

d i

$$I = \int e^x \cos x \, dx$$

$u = e^x$ $\quad v' = \cos x$

$u' = e^x$ $\quad v = \sin x$

$$I = uv - \int vu'$$

$$I = e^x \sin x - \int e^x \sin x \, dx$$

$u = e^x$ $\quad v' = \sin x$

$u' = e^x$ $\quad v = -\cos x$

$$I = e^x \sin x - \left(-e^x \cos x - \int -e^x \cos x \, dx\right)$$

$$I = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

(Original function appears - replace with I)

$$I = e^x \sin x + e^x \cos x - I$$

$$2I = e^x(\sin x + \cos x)$$

$$I = \frac{1}{2}e^x(\sin x + \cos x) + C$$

Integration by Parts Worksheet Q13