Use of Logarithms in Integration

Recall that

$$\frac{d}{dx}\ln x = \frac{1}{x}, x > 0 .$$

The condition $x > 0$ is necessary as $\ln x$ is only defined when $x > 0$.

This means that

$$\int \frac{1}{x} dx = \ln x + C, x > 0$$

This result may be adapted slightly to find $\int \frac{1}{x} dx$ when $x < 0$.

The magnitude of a real number x is denoted by $|x|$ and is the positive numerical value of x, regardless of whether x itself is positive or negative.

It can be shown that:

$$\int \frac{1}{x} dx = \ln|x| + C \text{ for all non-zero values of } x.$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a}\ln|ax+b| + C$$

The magnitude signs can be omitted in practice if the logarithm of a positive number is involved.

Note that the second standard integral only applies when integrating the reciprocal of a linear function. It cannot be used to find the integrals such as $\int \frac{1}{x^2+1} dx$.

Ex 1

$$\int \frac{1}{x-2}\,dx$$

$$= \ln|x-2| + C$$

Ex 2

$$\int \frac{6}{1-3x}\,dx$$

$$= 6\int \frac{1}{1-3x}\,dx$$

$$= 6 \times -\frac{1}{3}\ln|1-3x| + C$$

$$= -2\ln|1-3x| + C$$

Ex 3

$$\int \frac{8}{2x+1}\,dx$$

$$= 8\int \frac{1}{2x+1}\,dx$$

$$= 8.\frac{1}{2}\ln|2x+1| + C$$

$$= 4\ln|2x+1| + C$$

Ex 4

$$\int\left(\frac{2}{x}+\frac{4}{6x+1}-\frac{6}{1-3x}\right)dx$$

$$=2\int\frac{1}{x}+4\int\frac{1}{6x+1}-6\int\frac{1}{1-3x}dx$$

$$=2\ln|x|+4.\frac{1}{6}\ln|6x+1|-6.\frac{1}{(-3)}\ln|1-3x|+C$$

$$=2\ln|x|+\frac{2}{3}\ln|6x+1|+2\ln|1-3x|+C$$

Reminder of Log Laws

$$\ln(ab)=\ln a+\ln b$$

$$\ln\left(\frac{a}{b}\right)=\ln a-\ln b$$

$$\ln a^{n}=n\ln a$$

Ex 5

Prove $\int_{1/2}^{7/2} \frac{1}{2x+1}\,dx = \ln 2$

$$\int_{1/2}^{7/2} \frac{1}{2x+1}\,dx$$

$$= \left[\frac{1}{2} \ln|2x+1| \right]_{1/2}^{7/2}$$

$$= \frac{1}{2}\ln|8| - \frac{1}{2}\ln|2|$$

$$= \ln 8^{1/2} - \ln 2^{1/2}$$

$$= \ln \frac{8^{1/2}}{2^{1/2}}$$

$$= \ln \frac{\sqrt{8}}{\sqrt{2}}$$

$$= \ln \frac{\sqrt{4}\sqrt{2}}{\sqrt{2}}$$

$$= \ln \frac{2\sqrt{2}}{\sqrt{2}}$$

$$= \underline{\underline{\ln 2}}$$

There are other ways of manipulating the logarithms in this question.

For example

$$\frac{1}{2}(\ln 8 - \ln 2) \qquad \text{or} \qquad \frac{1}{2}\ln(8) - \frac{1}{2}\ln(2)$$

$$\frac{1}{2}\ln\left(\frac{8}{2}\right) \qquad \qquad \frac{1}{2}\ln(2)^3 - \frac{1}{2}\ln 2$$

$$\frac{1}{2}\ln 4 \qquad \qquad \frac{3}{2}\ln 2 - \frac{1}{2}\ln 2$$

$$\ln 4^{\frac{1}{2}} \qquad \qquad \ln 2$$

$$\ln 2$$