Algebraic Skills

Comparison of Pascal's Triangle with Binomial Coefficients

row 0	$\binom{0}{0}$	1
row 1	$\binom{1}{0}$ $\binom{1}{1}$	1 1
row 2	$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$	1 2 1
row 3	$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$	1 3 3 1
row 4	$\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$	1 4 6 4 1

They are the same!!!

So, we can use binomial coefficients to expand $(x+y)^n$

Binomial Theorem

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{r}x^{n-r}y^r + \cdots + \binom{n}{n}y^n$$

A general term is given by $\binom{n}{r}x^{n-r}y^r$

Binomial Theorem

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{r}x^{n-r}y^r + \cdots + \binom{n}{n}y^n$$

In your Exam formula sheet, this is written as:

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r \quad \text{where} \quad \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

The symbol $\sum$, called sigma, means take the sum of.

The equation $r = 0$ underneath this symbol means starting from $r = 0$.

The letter n at the top means until $r = n$.

Example 1

Find the coefficient of x^2y^5 in the expansion of $(x + y)^7$

General term $\binom{n}{r}x^{n-r}y^r \quad n = 7 \ \ x = x \ \ y = y$

$\binom{7}{r}x^{7-r}y^r$ so, $n = 7$ and $7 - r = 2, \ r = 5$

$\binom{7}{5}x^{7-5}y^5 = \binom{7}{5}x^2y^5$

Coefficient given by $\binom{7}{5} = \underline{\underline{21}}$

Example 2

Find the coefficient of x^2 in the expansion of $\left(x + \frac{2}{x}\right)^4$

General term $\binom{n}{r}x^{n-r}y^r \quad n = 4 \ \ x = x \ \ y = \frac{2}{x}$

$\binom{4}{r}x^{4-r}\left(\frac{2}{x}\right)^r = \binom{4}{r}x^{4-r}\left(2x^{-1}\right)^r$

$= \binom{4}{r}x^{4-r} \cdot x^{-r} \cdot 2^r$

$= \binom{4}{r}x^{4-2r}2^r$ So $n = 4$ and $4 - 2r = 2, \ \ r = 1$

$= 2\binom{4}{1}x^2$

Coefficient given by $2\binom{4}{1} = \underline{\underline{8}}$

Example 3

Find the term independent of x in the expansion of $\left(3x^2 + \frac{2}{x}\right)^9$

General term $\binom{n}{r}x^{n-r}y^r \quad n = 9 \ \ x = 3x^2 \ \ y = \frac{2}{x}$

$\binom{9}{r}(3x^2)^{9-r}\left(\frac{2}{x}\right)^r = \binom{9}{r}(3x^2)^{9-r}\left(2x^{-1}\right)^r$

$= \binom{9}{r}3^{9-r} \cdot 2^r \cdot x^{18-2r-r}$

$$= \binom{9}{r} 3^{9-r} \cdot 2^r \cdot x^{18-3r}$$

This term is independent of x if $18 - 3r = 0 \Rightarrow r = 6$

$$= \binom{9}{6} 3^3 \cdot 2^6$$

$$= 84 \times 27 \times 64$$

$$= \underline{145\,152}$$

Binomial Theorem Worksheet Q9 - Q15

2012 Q4

Write down and simplify the general term in the expansion of $\left(2x - \frac{1}{x^2}\right)^9$. **3**

Hence, or otherwise, obtain the term independent of x. **2**

General term $\binom{9}{r}(2x)^{9-r}\left(-\frac{1}{x^2}\right)^r$

$$= \binom{9}{r} \times 2^{9-r} \times x^{9-r} \times -\frac{1^r}{x^{2r}}$$

$$= \binom{9}{r} \times (-1)^r\, 2^{9-r}\, x^{9-3r}$$ •³ 'simplify 'x' terms'

The term independent of x occurs when:

$9 - 3r = 0$ i.e. $r = 3$

The term is : $\frac{9!}{6!\,3!}(-1)^3\, 2^6$

$$= \underline{\underline{-5376}}$$

2008 Q8

Write down and simplify the general term in the expansion of $\left(x^2 + \frac{1}{x}\right)^{10}$. **3**

Hence, or otherwise, obtain the term in x^{14}. **2**

General term $\binom{10}{r}(x^2)^{10-r}(x^{-1})^r$

$$= \binom{10}{r} x^{20-2r} x^{-r}$$

$$= \binom{10}{r} x^{20-3r}$$

We want the x^{14} term. Hence $20 - 3r = 14$

$\Rightarrow r = 2$

Using $\binom{10}{r} x^{20-3r}$ the term is $\binom{10}{2} x^{20-3(2)}$

$$= \underline{45x^{14}}$$