Differential Equations (1)

A differential equation is an equation connecting x, y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

The order of the differential equation is the value of the largest differential.

For example 1) $y\frac{dy}{dx} + 2xy = x$ 2) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 4 = \sin x$

1) is of the **first order** and 2) is of the **second order**

In order to solve a differential equation in the variables x and y, it is necessary to find some function $y = f(x)$ which satisfies the equation.

Let us consider the first order differential equation $\frac{dy}{dx} = 2x + 3$.

We know, by integration, that solutions to this differential equation will be of the form $y = x^2 + 3x + C$. This is called a **general solution**.

A **particular solution** is found by choosing values of x and y.

First Order Differential Equations with Variables Separable

Any first order differential equation that can be expressed in the form

$$f(y)\frac{dy}{dx} = g(x)$$

can be solved by separating the variables **(variable separable)**

Suppose

$$f(y)\frac{dy}{dx} = g(x)$$

$$\int f(y)dy = \int g(x)dx$$

Ex 1 Find the general solution of $y\frac{dy}{dx} = \frac{1}{x^2}$ $(\times dx)$

$y\,dy = \frac{1}{x^2}dx$ The variables have been separated

$\int y\,dy = \int \frac{1}{x^2}dx$

$\frac{1}{2}y^2 = -\frac{1}{x} + C$

$y^2 = -\frac{2}{x} + C$ (Normally we will want to write y explicitly)

Ex 2 Find the general solution of $(x+2)\frac{dy}{dx} = 1$ $\times dx$ $\div (x+2)$

$dy = \frac{dx}{x+2}$ The variables have been separated

$\int dy = \int \frac{1}{x+2}dx$

$y = \ln(x+2) + C$

Ex 3 Show that the general solution of $x^2\frac{dy}{dx} = y + 3$

can be written as $y = Ae^{-\frac{1}{x}} - 3$

$x^2 dy = (y+3)dx$

$\frac{dy}{y+3} = \frac{dx}{x^2}$

$\int \frac{1}{y+3}dy = \int \frac{1}{x^2}dx$

$\ln(y+3) = -\frac{1}{x} + C$

$y + 3 = e^{-\frac{1}{x}+C}$

$y = e^{-\frac{1}{x}+C} - 3$ let $A = e^C$

$y = Ae^{-\frac{1}{x}} - 3$

Ex 4 Find the general solution of $\frac{dy}{dx} = 4y$

Express y explicitly in terms of x.

$\frac{dy}{dx} = 4y$

$dy = 4y\,dx$

$\frac{1}{y}dy = 4dx$ The variables have been separated

$\int \frac{1}{y}dy = \int 4dx$

$\ln y = 4x + C$ $(e^{\wedge})$

$y = e^{4x+C}$

$y = e^{4x}.e^{c}$

$y = Ae^{4x}$ (Where $A = e^C$)

Ex 5 Find the general solution of $e^{4y}\dfrac{dy}{dx} - x = 2$

Express y explicitly in terms of x

$$e^{4y}\frac{dy}{dx} - x = 2$$

$$e^{4y}\frac{dy}{dx} = x + 2 \qquad \times\, dx$$

$$e^{4y}dy = (x+2)dx$$ The variables have been separated

$$\int e^{4y}dy = \int (x+2)dx$$

$$\frac{1}{4}e^{4y} = \frac{1}{2}x^2 + 2x + C \qquad (\times 4)$$

$$e^{4y} = 2x^2 + 8x + C \qquad (\ln)$$

$$4y = \ln(2x^2 + 8x + C)$$

$$y = \frac{1}{4}\ln(2x^2 + 8x + C)$$

Ex 6 Find the general solution of $\dfrac{dy}{dx} = 2x\sqrt{9-y^2}$

Express y explicitly in terms of x

$$\frac{dy}{dx} = 2x\sqrt{9-y^2} \qquad \times\, dx$$

$$dy = 2x\sqrt{9-y^2}\,dx \qquad \div \sqrt{9-y^2}$$

$$\frac{1}{\sqrt{9-y^2}}dy = 2x\,dx$$ The variables have been separated

$$\int \frac{1}{\sqrt{9-y^2}}dy = \int 2x\,dx$$

$$\sin^{-1}\left(\frac{y}{3}\right) = x^2 + C \qquad (\sin)$$

$$\frac{y}{3} = \sin(x^2 + C)$$

$$y = 3\sin(x^2 + C)$$

Ex 7 (a) Express $\frac{2x+3}{x(x+1)}$ in partial fractions

(b) Hence find the general solution of $x(x+1)\frac{dy}{dx}=y(2x+3)$ expressing y in terms of x.

(a) It can be shown that $\frac{2x+3}{x(x+1)}=\frac{3}{x}-\frac{1}{x+1}$

(b)

$$x(x+1)\frac{dy}{dx}=y(2x+3)$$

$$x(x+1)dy=y(2x+3)dx$$

$$\frac{1}{y}dy=\frac{2x+3}{x(x+1)}dx$$

$$\int\frac{1}{y}dy=\int\frac{2x+3}{x(x+1)}$$

$$\int\frac{1}{y}dy=\int\left(\frac{3}{x}-\frac{1}{x+1}\right)dx$$

$$\ln y=3\ln x-\ln(x+1)+C$$

$$\ln y=\ln\left(x^3\right)-\ln(x+1)+C$$

$$\ln y=\ln\left(\frac{x^3}{x+1}\right)+C \quad \left(e^\wedge\right)$$

$$y=\left(\frac{x^3}{x+1}\right).e^c$$

$$y=\frac{Ax^3}{x+1} \qquad \left(\text{Where } A=e^c\right)$$

Particular Solutions of Differential Equations

Ex 1 Find the particular solution of $\frac{dy}{dx}=x(y-2)$, given $x=0$ when $y=5$.

$$\frac{dy}{y-2}=xdx$$

The variables have been separated

$$\int\frac{dy}{y-2}=\int xdx$$

$$\ln(y-2)=\frac{1}{2}x^2+C$$

When $x=0$, $y=5$ then $C=\ln 3$

$$\ln(y-2) = \frac{1}{2}x^2 + \ln 3$$

$$\ln(y-2) - \ln 3 = \frac{1}{2}x^2$$

$$\ln\left(\frac{y-2}{3}\right) = \frac{1}{2}x^2 \qquad (e^\wedge)$$

$$\frac{y-2}{3} = e^{\frac{1}{2}x^2}$$

$$y = 3e^{\frac{1}{2}x^2} + 2$$

2012 Q15

Marks

(*a*) Express $\frac{1}{(x-1)(x+2)^2}$ in partial fractions. **4**

(*b*) Obtain the general solution of the differential equation

$$(x-1)\frac{dy}{dx} - y = \frac{x-1}{(x+2)^2},$$

expressing your answer in the form $y = f(x)$. **7**

2013 Q16

In an environment without enough resources to support a population greater than 1000, the population $P(t)$ at time t is governed by Verhurst's law

$$\frac{dP}{dt} = P(1000 - P).$$

Show that

$$\ln\frac{P}{1000-P} = 1000t + C \quad \text{for some constant } C.$$ **4**

Hence show that

$$P(t) = \frac{1000K}{K + e^{-1000t}} \quad \text{for some constant } K.$$ **3**

Given that $P(0) = 200$, determine at what time t, $P(t) = 900$. **3**