Proofs

Proof by Induction (2)

Induction is also used in proving summation examples.

When using induction in such examples it is useful to note that:

$$\sum_{r=1}^{k+1} f(r) = \sum_{r=1}^{k} f(r) + f(k+1)$$

Example 1

Prove that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ where $\forall n \in N$

Prove true for n = 1

LHS : $\sum_{r=1}^{n} r = \sum_{r=1}^{1} r = 1$ RHS : $\frac{1}{2}(1)(1+1) = 1$

LHS = RHS

$\therefore$ true for n = 1

Assume true for n = k $\Rightarrow \sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$

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Prove true for n = $k+1$

We want.... $\sum_{r=1}^{k+1} r = \frac{1}{2}(k+1)((k+1)+1)$

$$\sum_{r=1}^{k+1} r = \sum_{r=1}^{k} r + (k+1)$$

$$= \frac{1}{2}(k)(k+1) + (k+1)$$

$$= (k+1)\left(\frac{1}{2}k + 1\right)$$

$$= \frac{1}{2}(k+1)(k+2)$$

$$= \frac{1}{2}(k+1)((k+1)+1)$$

Hence if true for n = k, it is also true for n = $k+1$.

Since true for n = 1 then by PMI $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1) \quad \forall n \in N$

Example 2

Prove that $\sum_{r=1}^{n} (-1)^{r-1} r^2 = \frac{1}{2}(-1)^{n-1} n(n+1) \quad \forall n \in N$

Prove true for n = 1

LHS : $(-1)^{1-1} \times 1^2 = 1$

RHS : $\frac{1}{2}(-1)^{1-1} \times 1(1+1)$

$= \frac{1}{2} \times 2$

$= 1$

LHS = RHS

$\therefore$ true for n = 1

Assume true for n = k $\Rightarrow \sum_{r=1}^{k} (-1)^{r-1} r^2 = \frac{1}{2}(-1)^{k-1} k(k+1)$

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Prove true for n = $k+1$

We want.... $\sum_{r=1}^{k+1} (-1)^{r-1} r^2 = \frac{1}{2}(-1)^{(k+1)-1}(k+1)((k+1)+1)$

$$\sum_{r=1}^{k+1} (-1)^{r-1} r^2 = \sum_{r=1}^{k} (-1)^{r-1} r^2 + (-1)^{(k+1)-1}(k+1)^2$$

$$= \frac{1}{2}(-1)^{k-1} k(k+1) + (-1)^k (k+1)^2$$

$$= (-1)^k (k+1)\left(\frac{1}{2}(-1)^{-1} k + (k+1)\right)$$

$$= \frac{1}{2}(-1)^k (k+1)((-1)^{-1} k + 2(k+1))$$

$$= \frac{1}{2}(-1)^k (k+1)(-k + 2k + 2)$$

$$= \frac{1}{2}(-1)^k (k+1)(k+2)$$

$$= \frac{1}{2}(-1)^{(k+1)-1}(k+1)((k+1)+1)$$

$$= \frac{1}{2}(-1)^{(k+1)-1}(k+1)((k+1)+1)$$

Hence if true for n = k, it is also true for n = k + 1.

Since true for n = 1 then by PMI

$$\sum_{r=1}^{n}(-1)^{r-1}r^2 = \frac{1}{2}(-1)^{n-1}n(n+1) \quad \forall n \in N$$

Example 3

Prove that $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} \quad \forall n \in \mathbb{Z}^+$

Prove true for n = 1

LHS : $1^3 = 1$ RHS : $\frac{1^2(1+1)^2}{4} = 1$

LHS = RHS

∴ true for n = 1

Assume true for n = k ⇒ $\sum_{r=1}^{k} r^3 = \frac{k^2(k+1)^2}{4}$

Assume true for n = k ⇒ $\sum_{r=1}^{k} r^3 = \frac{k^2(k+1)^2}{4}$

Prove true for n = k + 1

We want.... $\sum_{r=1}^{k+1} r^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$

$$\sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$= \frac{(k+1)^2(k^2 + 4(k+1))}{4}$$

$$= \frac{(k+1)^2(k^2 + 4k + 4)}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= \frac{(k+1)^2((k+1)+1)^2}{4}$$

$$= \frac{(k+1)^2((k+1)+1)^2}{4}$$

Hence if true for n = k, it is also true for n = $k + 1$.

Since true for n = 1 then by PMI

$$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} \quad \forall n \in \mathbb{Z}^+$$