Integration Techniques

Recall the Standard Integrals:

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

- Using Division to help Integrate Remember $Q + \frac{R}{D}$
- Using Partial Fractions to Integrate
- A Combination of Division and Partial Fractions

Ex 1 $\displaystyle\int \frac{x^2 - x + 3}{x-1}\,dx$

$$\int \frac{x^2 - x + 3}{x-1}\,dx = \int x + \frac{3}{x-1}\,dx$$

$$= \frac{1}{2}x^2 + 3\ln|x-1| + C$$

Ex 2 (a) Express $\dfrac{1}{x^2-x-6}$ in partial fractions.

(b) Hence find the <u>exact</u> value of $\displaystyle\int_0^1 \frac{1}{x^2-x-6}\,dx$

giving your answer in the form $k\ln a$.

(a) $$\frac{1}{x^2-x-6}=\frac{\frac{1}{5}}{x-3}-\frac{\frac{1}{5}}{x+2}=\underline{\underline{\frac{1}{5(x-3)}-\frac{1}{5(x+2)}}}$$

(b) $$\int_0^1 \frac{1}{x^2-x-6}\,dx=\int_0^1\left(\frac{\frac{1}{5}}{x-3}-\frac{\frac{1}{5}}{x+2}\right)dx$$

$$=\left[\frac{1}{5}\ln|x-3|-\frac{1}{5}\ln|x+2|\right]_0^1$$

$$=\frac{1}{5}\Big[\big[\ln|-2|-\ln|3|\big]-\big[\ln|-3|-\ln|2|\big]\Big]$$

$$=\frac{1}{5}\Big[\big[\ln|2|-\ln|3|\big]-\big[\ln|3|-\ln|2|\big]\Big]$$

$$= \frac{1}{5}\left[2\ln|2| - 2\ln|3|\right]$$

$$= \frac{2}{5}\left[\ln|2| - \ln|3|\right]$$

$$= \frac{2}{5}\ln\left(\frac{2}{3}\right)$$

Ex 3 (a) Express $\dfrac{x}{x^2-1}$ in partial fractions.

(b) Hence find $\displaystyle\int \frac{x^3}{x^2-1}\,dx$

(a) $$\frac{x}{x^2-1} = \frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x+1} = \frac{1}{2(x-1)} + \frac{1}{2(x+1)}$$

(b) Note that $\dfrac{x^3}{x^2-1}$ is an improper rational function and algebraic long division must be used before integration.

$$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$$

$$= x + \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1} \quad \text{(the partial fractions from (a))}$$

Hence $\displaystyle\int \frac{x^3}{x^2-1}dx = \int \left(x + \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1} \right) dx$

$$= \frac{1}{2}x^2 + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$