1st Order Differential Eqns using the IF

These are equations of the form $\frac{dy}{dx} + p(x).y = q(x)$ (*)

Example

Solve the DE $\frac{dy}{dx} - 3x^2 y = 0$

We could use separation of variables but here is an alternative method:

multiply the equation by e^{-x^3}

$$e^{-x^3}\frac{dy}{dx} - 3x^2 e^{-x^3} y = 0$$

This can be written as $\frac{d}{dx}\left(e^{-x^3}.y\right) = 0$

[Check using Product and Chain Rules]

Therefore $e^{-x^3}.y = c$ (c = arbitrary constant)

Therefore $y = ce^{x^3}$

Why e^{-x^3}? In general, for (*), we let $\mu(x) = \int p(x)dx$ and we multiply both sides of (*) by the "Integrating Factor" $e^{\mu(x)}$

Then (*) becomes

$$e^{\mu(x)}\frac{dy}{dx} + p(x)e^{\mu(x)}y = e^{\mu(x)}.q(x)$$

Strategy

0. If necessary, divide the equation by the coefficient of $\frac{dy}{dx}$, so that this coefficient becomes 1.
1. Let $\mu(x) = \int p(x)dx$: then IF is $e^{\mu(x)}$
2. Multiply both sides of the equation by the IF
3. The LHS can now be written as $\frac{d}{dx}(e^{\mu x}.y)$
4. Integrate both sides
5. Solve for y.

Example 1

0. $\frac{dy}{dx} + 2y = e^x$ $\quad p(x) = 2$

1. Here $\mu(x) = \int 2dx = 2x$; so the IF is e^{2x}

2. Multiplying the equation we get $e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{3x}$

3. i.e. $\frac{d}{dx}(e^{2x}.y) = e^{3x}$

4. Integrating, we obtain $e^{2x}.y = \int e^{3x}dx$

$$= \frac{1}{3}e^{3x} + c$$

5. $\therefore y = \frac{1}{3}e^x + ce^{-2x}$

Example 2 $x\frac{dy}{dx} - y = x^3cosx$

0. $\frac{dy}{dx} - \frac{1}{x}.y = x^2cosx \quad p(x) = -\frac{1}{x}$

1. $\mu(x) = \int \left(-\frac{1}{x}\right) dx = -lnx$; so the IF is $e^{-lnx} = e^{ln\frac{1}{x}} = \frac{1}{x}$

2. $\frac{1}{x}.\frac{dy}{dx} - \frac{1}{x^2}.y = xcosx$

3. $\frac{d}{dx}\left(\frac{1}{x}.y\right) = xcosx$

4. $\frac{1}{x}.y = \int xcosxdx$ (int by parts)

$= xsinx - \int sinxdx$

$= xsinx + cosx + c$

5. $\therefore y = x^2sinx + xcosx + cx$

Exercise

Find the general solution of each of the following:

1) $x\frac{dy}{dx} + (x-2)y = x^3$

2) $\frac{dy}{dx} - 2y = 6e^{-x}$

3) $(1+x^2)\frac{dy}{dx} - xy = x(1+x^2)$

Find the particular solution of

4) $\frac{dy}{dx} + y = 2x + 4$ given $y = 1\ when\ x = 0$