Sequences and Series

Geometric Series

$$S_n = \frac{a(1-r^n)}{1-r};\ r \neq 1 \qquad \text{use this when r } < 1$$

or

$$S_n = \frac{a(r^n-1)}{r-1},\ r \neq 1 \qquad \text{use this when r } > 1$$

Example 1

Find the sum of the first 9 terms of the geometric series

$$4 + 8 + 16 + 32 + \ldots$$

$a = 4$, $r = 2$ and $n = 9$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_9 = \frac{4(2^9 - 1)}{2 - 1}$$

$$S_9 = 4(2^9 - 1)$$

$$S_9 = 2044$$

Example 2

Evaluate $\sum_{k=1}^{20} 0.9^k$ giving your answer correct to 3 d.p.

$k = 1 \quad k = 2 \quad \ldots \quad k = 20$

$0.9^1 \quad 0.9^2 \quad \ldots \quad 0.9^{20} \qquad a = 0.9,\ r = 0.9$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$= \frac{0.9(1 - 0.9^{20})}{1 - 0.9}$$

$$= 7.906\ (3d.p)$$

Example 3

Evaluate the sum of the geometric series

$$4 + 20 + 100 + \dots + 62500$$

$$a = 4, \quad r = 5$$

We must determine how many terms are in the series .

We know that $u_n = 625000$ $\Rightarrow$

$$ar^{n-1} = 62500$$
$$4 \times 5^{n-1} = 62500$$
$$5^{n-1} = 15625$$
$$ln5^{n-1} = ln15625$$
$$(n-1)ln5 = ln15625$$
$$n-1 = \frac{ln15625}{ln5}$$
$$n = 7$$

There are 7 terms in the series.

$$a = 4, \quad r = 5, n = 7$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$= \frac{4(5^7 - 1)}{5 - 1}$$
$$= 78124$$

Example 4

Find the sum of the geometric series

$$6 + 12 + 24 + 48 + \dots + 49\,152$$

$a = 6$, r = 2 and n =?

We must determine how many terms are in the series.

We know that $U_n = 49\,152$

$$u_n = ar^{n-1}$$
$$49152 = 6 \times 2^{n-1}$$
$$2^{n-1} = \frac{49152}{6}$$
$$2^{n-1} = 8192$$
$$\ln(2^{n-1}) = \ln 8192$$
$$(n-1)\ln 2 = \ln 8192$$
$$n - 1 = \frac{\ln 8192}{\ln 2}$$
$$n - 1 = 13$$
$$n = 14$$

There are 14 terms in the series.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{14} = \frac{6(2^{14} - 1)}{2 - 1} = \underline{\underline{98298}}$$

Example 4

Find the least number of terms of the geometric series 4 + 12 + 36 + 108 + ... which must be added to give a sum exceeding 1 000 000.

$$a = 4, \quad r = 3$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{4(3^n - 1)}{3 - 1}$$

$$= 2(3^n - 1)$$

$$2(3^n - 1) > 1000000$$

$$3^n - 1 > 500000$$

$$3^n > 500001$$

$$ln3^n > ln500001$$

$$nln3 > ln500001$$

$$n > \frac{ln500001}{ln3}$$

$$n > 11.94$$

This means that at least 12 terms must be added to give a sum exceeding 1 000 000.

Example 6

Find the geometric series 8 + 24 + 72 + 216 + ...

Find the least number of terms which much be taken to give a sum exceeding 1, 000, 000.

$a = 8$, $r = 3$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{8(3^n - 1)}{3 - 1}$$

$$= 4(3^n - 1)$$

Set S_n = 1 000 000

$1\,000\,000 = 4(3^n - 1)$
$250000 = 3^n - 1$
$250001 = 3^n$

$\ln(3^n) = \ln 250001$

$n\ln 3 = \ln 250001$

$$n = \frac{\ln 250001}{\ln 3}$$

$n = 11.31\ldots\ldots$

So we need at least 12 terms.

Check

$S_n = 4(3^n - 1)$
$S_{11} = 708\,584 < 1\,000\,000$
$S_{12} = 2\,125\,760 > 1\,000\,000$

Exercise

A geometric sequence has first term 1 and common ratio 4.

Find the smallest value of n for which $S_n > 2649$. **$r = -\frac{1}{7}$**

A geometric sequence has ratio $\frac{1}{5}$ and $S_3 = \frac{1}{25}$.

Find the first term. **$a = \frac{1}{31}$**