The Inverse

9 Dividing

And what about division? Well we **don't** actually divide matrices, we do it this way:

$$A \div B$$
$$= A \times \frac{1}{B}$$
$$= A \times B^{-1}$$

where $\boldsymbol{B^{-1}}$ means the "inverse" of B.

We'll learn more about the Inverse of a Matrix later.

10 Identity Matrix

The "Identity Matrix" is the matrix equivalent to the number "1":

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- It is "square" (has same number of rows and columns),
- It has **1**'s on the diagonal and **0**s everywhere else,
- It's symbol is the capital letter **I**.

It is a **special matrix** , because when you multiply by it, the original is unchanged:

$$\mathbf{A \times I = A}$$
$$\mathbf{I \times A = A}$$

The Inverse of a 2×2 matrix

If $AC = I$ then A is the inverse of C (& vice versa).

Example

Show that $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$ are inverses of each other.

+ +

+ +

= ∴ inverses of each other.

Generally, inverse of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

where *ad-bc* is the *det(A).*

Note/

If *det(A)* = 0, *A* does not have an inverse (i.e. singular matrix).

Example 1

Given that $A = \begin{pmatrix} 4 & 3 \\ 2 & 2 \end{pmatrix}$, find A^{-1}, if it exists.

det $A = (ad - bc) = (4 \times 2) - (2 \times 3) = 8 - 6 = 2 \neq 0$, so A^{-1} exist.

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -3 \\ -2 & 4 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & \frac{-3}{2} \\ -1 & 2 \end{pmatrix}}}$$

Example 2

Let $A = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$

(a) Show that the matrix A is invertible.

(b) Show that $A^2 = 5A - 2I$

(c) Hence show that $A^{-1} = \frac{1}{2}(5I - A)$ and obtain the matrix A^{-1}

(a) $\det A = (ad - bc) = (2\times 3) - (4\times 1) = 2$

$\det A \neq 0$, so the matrix A is invertible.

(b) $A^2 = AA = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$

$= \begin{pmatrix} 8 & 20 \\ 5 & 13 \end{pmatrix}$

(c) $$A^2 = 5A - 2I$$

($\times A^{-1}$) $$A^{-1} \cdot A^2 = A^{-1}(5A - 2I)$$

$$A = 5A^{-1} \cdot A - 2A^{-1} \cdot I$$

$$A = 5I - 2A^{-1}$$

$$2A^{-1} = 5I - A$$

$$\underline{\underline{A^{-1} = \frac{1}{2}(5I - A) \text{ as req}^{\text{d}}}}$$

Now

$$5I - A = 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}}}$$

The Inverse of a 3 × 3 Matrix

Ex 1 Find the inverse of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \qquad \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$$R_3 \to R_3 - 2R_1 \quad \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right)$$

$$R_2 \to R_2 - R_1 \quad \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right)$$

$$R_3 \to 2R_3 - R_2 \quad \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -7 & -3 & -1 & 2 \end{array}\right)$$

$$R_1 \to 7R_1 + R_3 \quad \left(\begin{array}{ccc|ccc} 7 & 7 & 0 & 4 & -1 & 2 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -7 & -3 & -1 & 2 \end{array}\right)$$

$$R_2 \to 7R_2 + R_3 \quad \left(\begin{array}{ccc|ccc} 7 & 7 & 0 & 4 & -1 & 2 \\ 0 & -14 & 0 & -10 & 6 & 2 \\ 0 & 0 & -7 & -3 & -1 & 2 \end{array}\right)$$

$$R_1 \to 2R_1 + R_2 \quad \left(\begin{array}{ccc|ccc} 14 & 0 & 0 & -2 & -4 & 6 \\ 0 & -14 & 0 & -10 & 6 & 2 \\ 0 & 0 & -7 & -3 & -1 & 2 \end{array}\right)$$

$$R_1 \to \frac{1}{14} R_1$$
$$R_2 \to -\frac{1}{14} R_2$$
$$R_3 \to -\frac{1}{7} R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{7} & -\frac{2}{7} & \frac{3}{7} \\ 0 & 1 & 0 & \frac{5}{7} & -\frac{3}{7} & -\frac{1}{7} \\ 0 & 0 & 1 & \frac{3}{7} & \frac{1}{7} & \frac{2}{7} \end{array}\right)$$

$$A^{-1} = \frac{1}{7}\begin{pmatrix} -1 & 2 & 3 \\ 5 & -3 & -1 \\ 3 & 1 & -2 \end{pmatrix}$$