The Identity Matrix

"Show that ...."

10 Identity Matrix

The "Identity Matrix" is the matrix equivalent to the number "1":

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- It is "square" (has same number of rows and columns),
- It has **1**'s on the diagonal and **0**s everywhere else,
- It's symbol is the capital letter **I**.

It is a **special matrix** , because when you multiply by it, the original is unchanged:

A × I = A

I × A = A

Example ***(2012 Q9)***

A non-singular $n \times n$ matrix A satisfies the equation $A + A^{-1} = I$, where I is the $n \times n$ identity matrix.

Show that $A^3 = kI$ and state the value of k.

$$A + A^{-1} = I$$

$(\times A)$ $$A^2 + I = A$$

using $A + A^{-1} = 1$
$A = I - A^{-1}$

$$A^2 + I = I - A^{-1}$$

$$A^2 = -A^{-1}$$

$(\times A)$ $$A^3 = -I$$

i.e. $\underline{\underline{k = -1}}$

Example ***(2005 Q7)***

Given the matrix $A = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$, show that $A^2 + A = kI$ for some constant k, where I is the 3×3 unit matrix.

Obtain the values of p and q for which $A^{-1} = pA + qI$.

$$A^2 + A = kI$$

$$A^2 + A = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}\begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} + \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -4 & -2 \\ -1 & 2 & -1 \\ 1 & 2 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$= 2I$ i.e. $\underline{\underline{k = 2}}$

$A^2 + A = kI$

$A^{-1} = pA + qI$

$$A^2 + A = 2I$$

$$A^{-1}(A^2 + A) = 2A^{-1}I$$

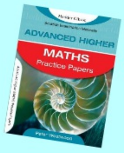
$$A^{-1}(A^2 + A) = 2A^{-1}$$

$$A^{-1}A^2 + A^{-1}A = 2A^{-1}$$

$$A^1 + I = 2A^{-1}$$

$$A^{-1} = \frac{1}{2}A + \frac{1}{2}I$$

i.e. $p = \frac{1}{2}$ and $q = \frac{1}{2}$



The matrix $A = \begin{pmatrix} 2 & -3 \\ 4 & 5 \end{pmatrix}$.

(a) Show that $A^2 = 7A - 22I$.

(b) Hence show that:

(i) $A^3 = 27A - 154I$ (without evaluating A^3);

(ii) $A^{-1} = \frac{1}{22}(7I - A)$ (without evaluating A^{-1}).

(a) $A^2 = \begin{pmatrix} -8 & -21 \\ 28 & 13 \end{pmatrix}$ $\quad 7A - 22I = \begin{pmatrix} -8 & -21 \\ 28 & 13 \end{pmatrix}$

$\therefore \underline{\underline{A^2 = 7A - 22I \text{ as req}^d}}$

(b) (i) $A^2 = 7A - 22I$

$(\times A)$ $A^3 = A(7A - 22I)$

$A^3 = 7A^2 - 22AI$

$A^3 = 7A^2 - 22A$

$A^3 = 7(7A - 22I) - 22A$

$A^3 = 49A - 154I - 22A$

$\underline{\underline{A^3 = 27A - 154I}}$ as req[d]

(b) (ii) $A^2 = 7A - 22I$

$(\div A)$ $A = 7I - 22A^{-1}$

$$A^{-1} = \frac{1}{22}(7I - A)$$