Integration by Substitution

Consider $\int x(x^2+3)^4 dx$.

Our only method to deal with this integration is to multiply out the brackets. Not a pleasant prospect so we need to use a method called substitution.

Integration by substitution involves rewriting the entire integral (including the "*dx*" and any limits) in terms of another variable before integrating.

Ex 1. $\int x(x^2+3)^4 dx$ using the substitution $u = x^2 + 3$

$$u = x^2 + 3$$

$$\frac{du}{dx} = 2x$$

Why? because the derivative is a multiple of something in the product.

$$du = 2xdx$$

$$\tfrac{1}{2}du = xdx$$

This means that xdx can be replaced by $\tfrac{1}{2}du$

and the term $(x^2+3)^4$ can be replaced by u^4

$$\int x\left(x^2+3\right)^4 dx = \frac{1}{2}\int u^4 du$$

$$= \frac{1}{2}\left(\frac{u^5}{5}\right)+C$$

$$= \frac{u^5}{10}+C \quad \textit{replace } u = x^2+3$$

$$= \frac{\left(x^2+3\right)^5}{10}+C \text{ or } \frac{1}{10}\left(x^2+3\right)^5+C$$

Note the answer for an indefinite integral must always be expressed in terms of the original variable.

Ex 2. $\int x^2\sqrt{1-x^3}\,dx$ using the substitution $u = 1-x^3$

$$u = 1-x^3$$

$$du = -3x^2dx$$

$$-\tfrac{1}{3}du = x^2dx$$

$$\int x^2\sqrt{1-x^3}\,dx = -\frac{1}{3}\int u^{\frac{1}{2}}du$$

$$= -\frac{1}{3}.\frac{2}{3}u^{\frac{3}{2}} + C$$

$$= -\frac{2}{9}u^{\frac{3}{2}} + C \quad \text{(replace u = } 1 - x^3\text{)}$$

$$= -\frac{2}{9}\left(1-x^3\right)^{\frac{3}{2}} + C$$

Ex 3. $\int \sin x\sqrt{1+\cos x}\,dx$ using the substitution $u = 1 + \cos x$

$$\int \sin x(1+\cos x)^{\frac{1}{2}}dx$$

$$u = 1 + cosx$$

$$du = -sinxdx$$

$$-du = sinxdx$$

$$\int \sin x(1+\cos x)^{\frac{1}{2}}dx \quad = -\int u^{\frac{1}{2}}du$$

$$= -\frac{2}{3}u^{\frac{3}{2}} + C \quad (replace\ u = 1 = \cos x)$$

$$= -\frac{2}{3}(1+\cos x)^{\frac{3}{2}} + C$$

Ex 4. $\int x(x-4)^6\,dx$ using the substitution $u = x - 4$.

$$u = x - 4.$$

$$\frac{du}{dx} = 1,$$

$$du = dx$$

$$\int x(x-4)^6\,dx \quad = \quad \int xu^6 du$$

We have to change x in terms of u.

Using $u = x - 4$ we make $x = u + 4$

$$\int x(x-4)^6\,dx = \int (u+4)u^6\,du$$
$$= \int u^7 + 4u^6\,du$$
$$= \frac{1}{8}u^8 + \frac{4}{7}u^7 + C$$
$$= \frac{1}{8}(x-4)^8 + \frac{4}{7}(x-4)^7 + C$$

Ex 5. Show that $\int \frac{1}{e^x+1}\,dx = x - \ln\left(e^x+1\right) + C$

by using the substitution $u = e^x$

$$u = e^x$$
$$du = e^x dx$$
$$\frac{1}{e^x}du = dx$$
$$\frac{1}{u}du = dx$$

can't replace *dx* with something involving *x*

$$\int \frac{1}{e^x + 1} dx \quad = \int \frac{1}{u+1} . \left(\frac{1}{u} du \right)$$

$$= \int \frac{1}{u(u+1)} du$$

We need to use partial fractions.

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

$$= \frac{A(u+1) + Bu}{u(u+1)}$$

$$1 = A(u + 1) + Bu$$

put u = -1 $\quad 1 = -B$

$\quad -1 = B$

put u = 0 $\quad 1 = A$

$$\int \frac{1}{e^x+1}\,dx = \int \frac{1}{u(u+1)}\,du = \int \frac{1}{u} - \frac{1}{u+1}\,du$$

$$= \ln u - \ln(u+1) + C$$

$$= \ln e^x - \ln(e^x+1) + C$$

$$= \underline{\underline{x - \ln(e^x+1) + C}}$$

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Q3a – 3 marks

(*a*) Use the substitution $t = x^4$ to obtain $\int \frac{x^3}{1+x^8}\,dx$. **3**

$t = x^4 \Rightarrow dt = 4x^3dx$	1	correct differential
$\int \frac{x^3}{1+x^8}dx = \frac{1}{4}\int \frac{4x^3}{1+(x^4)^2}dx$		
$= \frac{1}{4}\int \frac{1}{1+t^2}dt$	1	correct integral in t
$= \frac{1}{4}\tan^{-1} t + c$		
$= \frac{1}{4}\tan^{-1} x^4 + c$	1	correct answer