Transformation Matrices

Transformation Matrices

2 x 2 matrices can be associated with transformations of all points in a cartesian plane.

Transformations are defined as reflection in the x-axis, y-axis and in the line y = x, rotation, and dilatation (enlargement and reduction) or compositions of these.

Reflections

The coordinate (x, y) can be represented in matrix form by $\begin{pmatrix} x \\ y \end{pmatrix}$ or :

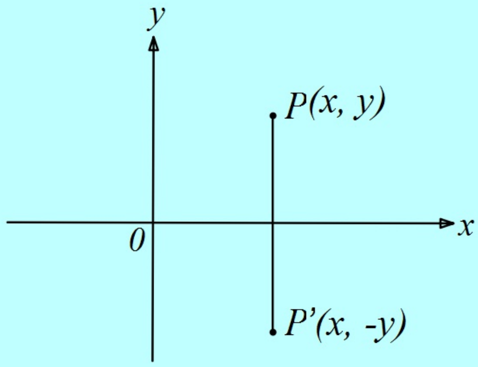
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

 since $\begin{pmatrix} 1x + 0y \\ 0x + 1y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

The following ∴ must also be true:

① Reflection in the x-axis

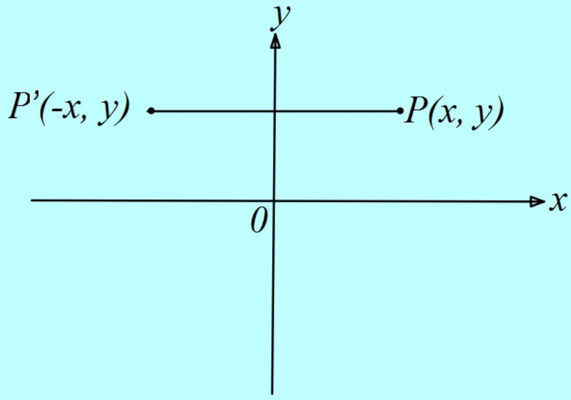
$$\begin{matrix} x & y \end{matrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



$$(x, y) \rightarrow (x, -y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

② Reflection in the y-axis

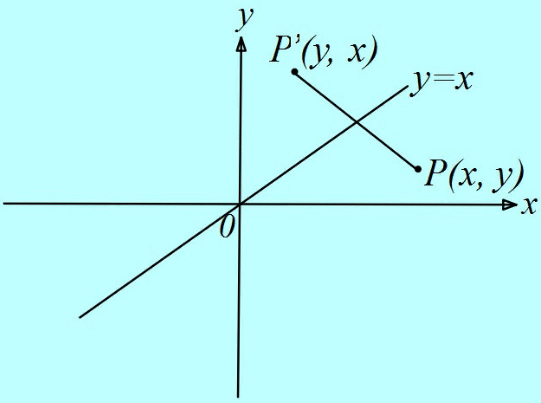
$$\begin{matrix} x & y \end{matrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



$$(x, y) \rightarrow (-x, y) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

③ Reflection in line y = x

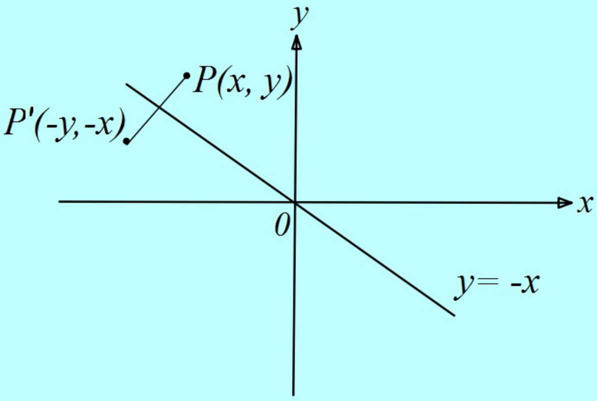
$$\begin{matrix} x & y \\ \end{matrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



$$(x, y) \rightarrow (y, x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

④ **Reflection in the line y = -x**

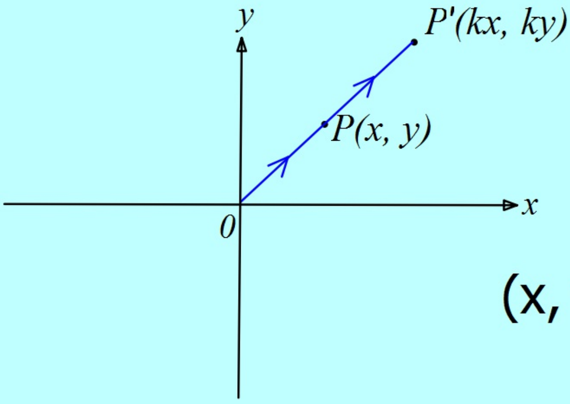
$$\begin{matrix} x & y \end{matrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



$$(x, y) \rightarrow (-y, -x) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

⑤ Stretch (k >1) or reduction (k <1)

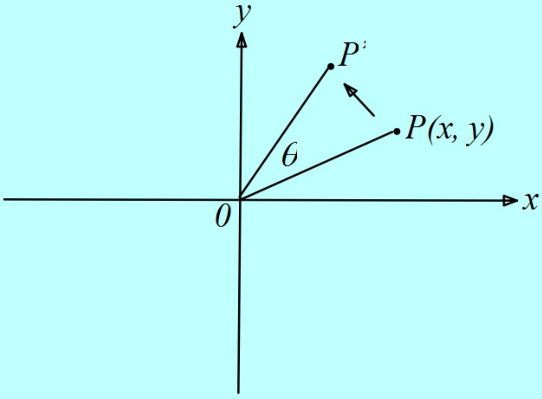
$$\begin{matrix} x & y \end{matrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



$$(x, y) \rightarrow (kx, ky) = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

<u>**Rotations**</u>

Suppose that a point P(x, y) is rotated through an angle of q about the origin in an <u>anticlockwise</u> direction as shown in the diagram below.



The 2 x 2 matrix associated with this rotation is:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

<u>YOU MUST REMEMBER THIS</u>

Example 1

Write down the 2 x 2 matrix associated with an anticlockwise rotation of $\frac{\pi}{2}$ radians about the origin.

Hence find the image of the point P(x, y) under this rotation.

$$\begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}}$$

$$\text{Now} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix},$$

hence the image of the point $P(x, y)$ is $\underline{\underline{P'(-y, x)}}$

Example 2

Write down the 2 x 2 matrix associated with an ***clockwise*** rotation of 30° about the origin.

Note that a ***clockwise*** rotation of 30° about the origin is equivalent to an ***anticlockwise*** rotation of 330° about the origin.

$$\begin{pmatrix} \cos 330^0 & -\sin 330^0 \\ \sin 330^0 & \cos 330^0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Exercise

Write down the 2 x 2 matrix associated with:

(i) a rotation of π about the origin.

(ii) a rotation of $\frac{\pi}{3}$ about the origin.

$$\begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Composite Transformations

If the matrices A, B represent transformations T_1, T_2 of a plane, then BA represents the transformation given by T_1 followed by T_2.

Note the order of the matrices.

DON'T FORGET

BA means A first

Example

The point $P(x, y)$ is given an anticlockwise rotation of $\frac{\pi}{2}$ radians about the origin and the image is then reflected in the x-axis.

(a) Find the matrix associated with this composite transformation.

(b) Find the coordinates of the image of the point $P(x, y)$ under this composite transformation.

Solution

(a) The matrix associated with an anticlockwise rotation of $\frac{\pi}{2}$ radians about the origin is

$$M_1 = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The matrix associated with reflection in the x-axis is

$$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The matrix associated with the composite transformation is

$$M_2M_1$$

$$M_2M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

(b) $$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}$$

Hence the image of the point $P(x, y)$ under the composite transformation is $P'\underline{\underline{(-y, -x)}}$.

Exercise

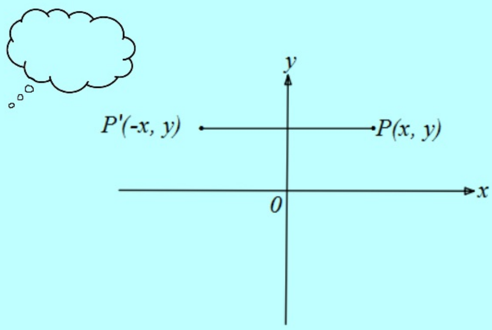
Write down the matrices A, B representing a reflection in the y-axis and an anti-clockwise rotation about the origin through 45°.

Hence obtain the matrices representing the result of applying:

(1) the reflection followed by the rotation
(2) the rotation followed by the reflection.

Reflection in y-axis (A)

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



Rotation through 45º (B)

$$B = \begin{pmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(1) **The reflection followed by the rotation**

A (reflection), B (rotation)

A is first so we write as BA

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

2 x ~~2~~ ~~2~~ x 2

$$= \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(2) **The rotation followed by the reflection**

B (rotation) A (reflection)

B is first so we write as AB

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$2 \times \cancel{2}$ $\cancel{2} \times 2$

$$= \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Note the resulting transformation are different.