Definite Integrals

You must be very careful when calculating definite integrals using substitution as the limits of integration change.

Ex 1. $\int_0^1 15x^2\left(x^3+1\right)^4 dx$

$$\text{Let } u = x^3 + 1$$
$$du = 3x^2 dx$$
$$\tfrac{1}{3}du = x^2 dx$$

$$\int_0^1 15x^2\left(x^3+1\right)^4 dx = \int_0^1 5u^4 du$$

New limits.

$u = x^3 + 1$

when $x = 0$, then $u = 0^3 + 1 = 1$

when $x = 1$, then $u = 1^3 + 1 = 2$

$$= \int_1^2 5u^4 du \qquad \text{(note new limits)}$$
$$= \left[u^5\right]_1^2$$
$$= (2)^5 - (1)^5$$
$$= \underline{\underline{31}}$$

Ex 2. $\int_0^{\pi/6} 10\sin^4 x \cos x dx$

$u = sinx$

$du = cosxdx$

$$\int_0^{\pi/6} 10\sin^4 x \cos x dx = \int_0^{\pi/6} 10u^4 du$$

New limits.

$u = \sin x$

when $x = 0$, then $u = \sin 0 = 0$

when $x = \pi/6$, then $u = \sin \pi/6 = ½$

$$= \int_0^{1/2} 10u^4 du \qquad \text{(note new limits)}$$

$$= \left[2u^5\right]_0^{1/2}$$

$$= (2\times(½)^5) - (2\times(0)^5)$$

$$= \frac{1}{16}$$

Ex3 Use the substitution $u = \cos x$ to show that

$$\int_0^{\frac{\pi}{3}} \tan x \, dx = \ln 2$$

$$\int_0^{\frac{\pi}{3}} \tan x \, dx = \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x} dx$$

$$u = \cos x$$

$$du = -\sin x dx$$

$$-du = \sin x dx$$

$$\int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x} dx = -\int_0^{\frac{\pi}{3}} \frac{1}{u} du$$

<u>New limits</u>.

$u = \cos x$

when $x = 0$, then $u = \cos 0 = 1$

when $x = \pi/3$, then $u = \cos \pi/3 = ½$

$$= -\int_1^{1/2} \frac{1}{u}\,du \qquad \text{(note new limits)}$$

$$= -\left[\ln u\right]_1^{1/2}$$

$$= -\left[\ln\left(\frac{1}{2}\right) - \ln(1)\right]$$

$$= -\ln\left(\frac{1}{2}\right)$$

$$= -\ln(2^{-1})$$

$$= \underline{\underline{\ln 2}}$$

Ex 4 Use the substitution $u = \sin x$ to evaluate $\int_0^{\frac{\pi}{2}} \cos^3 x\,dx$

$$\int_0^{\frac{\pi}{2}} \cos^3 x\,dx$$

$$u = sinx$$

$$du = cosxdx$$

$$\int_0^{\frac{\pi}{2}} \cos^3 x\, dx = \int_0^{\frac{\pi}{2}} \cos^2 x\, du$$

We must also replace the remaining term $\cos^2 x$ in the integrand.

$$\cos^2 x = 1 - \sin^2 x = 1 - u^2$$

New limits. when $x = 0$, then $u = \sin 0 = 0$

when $x = \frac{\pi}{2}$, then $u = \sin\frac{\pi}{2} = 1$

$$= \int_0^1 \left(1 - u^2\right) du$$

$$= \left[u - \frac{1}{3}u^3\right]_0^1$$

$$= \left(1 - \frac{1}{3}\right) - (0)$$

$$= \frac{2}{3}$$

Ex 5 Use the substitution $x = 2\sin\theta$ to show that

$$\int_0^1 \frac{x+1}{\sqrt{4-x^2}}dx = 2-\sqrt{3}+\frac{\pi}{6}$$

$$\int_0^1 \frac{x+1}{\sqrt{4-x^2}}dx$$

$$x = 2sin\theta$$

$$dx = 2cos\theta d\theta$$

<u>New Limits</u>

<u>When $x = 0$:</u>

$$2\sin\theta = 0$$

$$\sin\theta = 0$$

$$\theta = \sin^{-1}(0)$$

$$\theta = 0$$

<u>When $x = 1$:</u>

$$2\sin\theta = 1$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \sin^{-1}\left(\frac{1}{2}\right)$$

$$\theta = \frac{\pi}{6}$$

$$\int_0^1 \frac{x+1}{\sqrt{4-x^2}}dx = \frac{2sin\theta + 1}{\sqrt{4-4sin^2\theta}}$$

$$= \frac{2sin\theta + 1}{\sqrt{4(1-sin^2\theta)}}$$

$$= \frac{2sin\theta + 1}{\sqrt{4cos^2\theta}}$$

$$\int_0^1 \frac{x+1}{\sqrt{4-x^2}}dx = \int_0^{\frac{\pi}{6}} \frac{(2\sin\theta + 1)}{2\cos\theta}.2\cos\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} (2\sin\theta + 1)\, d\theta$$

$$= \left[-2\cos\theta + \theta\right]_0^{\frac{\pi}{6}}$$

$$= \left[-2\cos\frac{\pi}{6} + \frac{\pi}{6}\right] - \left[-2\cos 0 + 0\right]$$

$$= -2\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6} + 2(1)$$

$$= -\sqrt{3} + \frac{\pi}{6} + 2$$

$$= \underline{\underline{2 - \sqrt{3} + \frac{\pi}{6}}}$$