Implicit Differentiation

Consider the equation $x^2 + y^2 = 1$

$$y^2 = 1 - x^2$$

$$y = \pm\sqrt{1 - x^2}$$

This equation defines y **explicitly** as a function of x.

The equation $x^2 + y^2 = 1$ defines y **implicitly** in terms of x.

When y is defined **implicitly** in terms of x, it is possible to find an expression for $\frac{dy}{dx}$ without first expressing y explicitly in terms of x. This is useful, since it is often very difficult or impossible to express y explicitly in terms of x.

The derivative of y^2 with respect to x is found **using the chain rule** as follows:

$$\frac{d}{dx}y^2 = 2y \cdot \frac{dy}{dx}$$

So, $\frac{d}{dx}(y) = \frac{dy}{dx}$

$$\frac{d}{dx}\left(y^3\right) = 3y^2 \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}\sin(y) = cos y \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}e^y = e^y \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}\ln y = \frac{1}{y} \cdot \frac{dy}{dx}$$

Note that xy is actually a product of two functions of x (since y is a function of x) and the derivative of xy with respect to x is found using the product rule, as follows:

$$\frac{d}{dx}(xy) = x \cdot \frac{dy}{dx} + y \cdot 1$$

$$= x\frac{dy}{dx} + y$$

Example

The equation $x^2+y^2 = 4$ defines y implicitly in terms of x.

Find an expression for $\frac{dy}{dx}$ in terms of x and y.

Solution

$$x^2+y^2 = 4$$

Differentiate both sides of this equation with respect to x.

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$x + y\frac{dy}{dx} = 0$$

$$y\frac{dy}{dx} = -x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Example 2

The equation $\ln y = x + y$ defines y implicitly in terms of x.

Find an expression for $\frac{dy}{dx}$ in terms of x and y.

Solution

$$\ln y = x + y$$

Differentiate both sides of the equation with respect to x.

$$\frac{dy}{dx} \cdot \frac{1}{y} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = y + y\frac{dy}{dx}$$

$$\frac{dy}{dx} - y\frac{dy}{dx} = y$$

$$\frac{dy}{dx}(1 - y) = y$$

$$\frac{dy}{dx} = \frac{y}{1-y}$$

Example 3

The equation $x^2 + xy = 2$ defines y implicitly in terms of x.

Find an expression for $\frac{dy}{dx}$ in terms of x and y.

Solution

$$x^2 + xy = 2$$

$$2x + x \cdot \frac{dy}{dx} + y \cdot 1 = 0$$

$$x\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x-y}{x}$$

Example 4

The equation $y^2 - xy = x$ defines y implicitly in terms of x.

Find an expression for $\frac{dy}{dx}$ in terms of x and y.

Solution

$$y^2 - xy = x$$

$$2y \cdot \frac{dy}{dx} - x \cdot \frac{dy}{dx} + y \cdot (-1) = 1$$

$$\frac{dy}{dx}(2y - x) = 1 + y$$

$$\frac{dy}{dx} = \underline{\underline{\frac{1+y}{2y-x}}}$$

Example 5

The equation $x^2y^2 = x^4 + y^4$ defines y implicitly in terms of x.

Find an expression for $\frac{dy}{dx}$ in terms of x and y.

Solution

$$x^2y^2 = x^4 + y^4$$

$$x^2 . 2y . \frac{dy}{dx} + y^2 . 2x = 4x^3 + 4y^3 . \frac{dy}{dx}$$

$$2x^2y\frac{dy}{dx} + 2xy^2 = 4x^3 + 4y^3\frac{dy}{dx}$$

$$x^2y\frac{dy}{dx} + xy^2 = 2x^3 + 2y^3\frac{dy}{dx}$$

$$x^2y\frac{dy}{dx} - 2y^3\frac{dy}{dx} = 2x^3 - xy^2$$

$$\frac{dy}{dx}(x^2y - 2y^3) = 2x^3 - xy^2$$

$$\frac{dy}{dx} = \frac{2x^3 - xy^2}{x^2y - 2y^3}$$

Equations of Tangents

Example 1

A curve is defined by the implicit equation $x^2 + y^2 + 2x - 4y = 15$

Find the equation of the tangent at the point (3, 4) on the curve.

Solution

$$x^2 + y^2 + 2x - 4y = 15$$

$$2x + 2y . \frac{dy}{dx} + 2 - 4\frac{dy}{dx} = 0$$

$$x + y\frac{dy}{dx} + -2\frac{dy}{dx} = 0$$

$$y\frac{dy}{dx} - 2\frac{dy}{dx} = -x - 1$$

$$\frac{dy}{dx}(y - 2) = -x - 1$$

$$\frac{dy}{dx} = \frac{-x-1}{y-2}$$

At the point (3, 4)

$$\frac{dy}{dx} = \frac{-x-1}{y-2}$$

$$\frac{dy}{dx} = \frac{-3-1}{4-2}$$

$$\frac{dy}{dx} = -2$$

$$y - b = m(x - a)$$

$$y - 4 = -2(x - 3)$$

$$y - 4 = -2x + 6$$

$$y = -2x + 10$$

Example 2

A curve is defined by the implicit equation $2x^2 - 3xy - y^2 = 1$

Find the equation of the tangent at the point (2, 1) on the curve.

$$2x^2 - 3xy - y^2 = 1$$

$$4x - 3x \,.\, \frac{dy}{dx} - 3y - 2y\frac{dy}{dx} = 0$$

$$4x - 3y + \frac{dy}{dx}(-3x - 2y) = 0$$

$$\frac{dy}{dx}(-3x - 2y) = 3y - 4x$$

$$\frac{dy}{dx} = \frac{3y-4x}{-3x-2y}$$

At the point (2, 1) $$\frac{dy}{dx} = \frac{3y-4x}{-3x-2y}$$

$$\frac{dy}{dx} = \frac{3(1)-4(2)}{-3(2)-2(1)}$$

$$\frac{dy}{dx} = \frac{5}{8}$$

$$y - b = m(x - a)$$

$$y - 1 = \tfrac{5}{8}(x - 2)$$

$$8y - 8 = 5x - 10$$

$$\underline{\underline{5x - 8y - 2 = 0}}$$

Second Derivatives

The function $y = f(x)$ is defined implicitly by the equation $x^2 + 2xy = 1$

Find the expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x and y.

Solution

$$x^2 + 2xy = 1$$

$$2x + 2x\frac{dy}{dx} + 2y = 0$$

$$x + x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = \underline{\underline{\frac{-x-y}{x}}}$$

$$x + x\frac{dy}{dx} + y = 0 \quad \text{or} \quad \frac{dy}{dx} = \frac{-x-y}{x}$$

$$1 + x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$1 + x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

$$1 + x\frac{d^2y}{dx^2} + 2\left(\frac{-x-y}{x}\right) = 0$$

$$x + x^2\frac{d^2y}{dx^2} + 2(-x-y) = 0$$

$$x + x^2\frac{d^2y}{dx^2} - 2x - 2y = 0$$

$$x^2\frac{d^2y}{dx^2} = x + 2y$$

$$\frac{d^2y}{dx^2} = \frac{x+2y}{x^2}$$

Example 2

The function $y = f(x)$ is defined implicitly by the equation $y^2 - x^2 = 4$

Find the expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x and y.

Solution

$$y^2 - x^2 = 4$$

$$2y \cdot \frac{dy}{dx} - 2x = 0$$

$$y\frac{dy}{dx} - x = 0$$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$y\frac{dy}{dx} - x = 0 \qquad\qquad \frac{dy}{dx} = \frac{x}{y}$$

$$y\frac{dy}{dx} - x = 0$$

$$y \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{dy}{dx} - 1 = 0$$

$$y \cdot \frac{d^2y}{dx^2} + \frac{x}{y} \cdot \frac{x}{y} - 1 = 0$$

$$y \cdot \frac{d^2y}{dx^2} + \frac{x^2}{y^2} - 1 = 0$$

$$y^3 \cdot \frac{d^2y}{dx^2} + x^2 - y^2 = 0$$

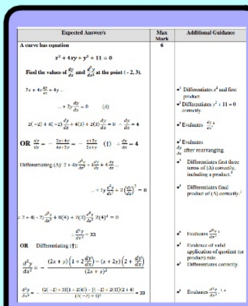
$$\frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3}$$

2013 Q11 (6 marks)

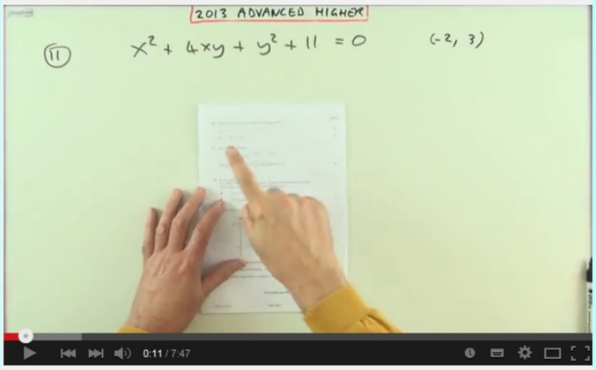
A curve has equation

$$x^2 + 4xy + y^2 + 11 = 0.$$

Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(-2, 3)$.



MARKING SCHEME



Logarithmic Differentiation

Recall the laws of logarithms below:

$$\ln(ab) = lna + lnb$$

$$ln\left(\frac{a}{b}\right) = lna - lnb$$

$$ln(a^n) = nlna$$

When a function involves the product of powers or roots, logarithmic differentiation can be used to find the derivative of the function.

Logarithmic differentiation involves taking the natural logarithm of the function before differentiating, the function should be simplified before differentiating.

Example 1

$y = 10^x$ use logarithmic differentiation to find $\frac{dy}{dx}$

$$y = 10^x$$

$$lny = ln(10^x)$$

$$lny = xln10$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = ln10$$

$$\frac{dy}{dx} = yln10$$

$$\frac{dy}{dx} = \underline{\underline{ln10 \times 10^x}}$$

Example 2

Given $y = 2^{3x}$ use logarithmic differentiation to find $\frac{dy}{dx}$

$$y = 2^{3x}$$

$$lny = ln2^{3x}$$

$$lny = 3xln2$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 3ln2$$

$$\frac{dy}{dx} = 3ln2 \times y$$

$$\frac{dy}{dx} = \underline{\underline{3ln2 \times 2^{3x}}}$$

Example 3

Given $y = \frac{x^2(x+1)^4}{\sqrt{4x+1}}$ use logarithmic differentiation to find $\frac{dy}{dx}$

$$y = \frac{x^2(x+1)^4}{\sqrt{4x+1}}$$

$$y = \frac{x^2(x+1)^4}{(4x+1)^{\frac{1}{2}}}$$

$$lny = ln\left(\frac{x^2(x+1)^4}{(4x+1)^{\frac{1}{2}}}\right)$$

$$lny = \ln\left(x^2(x+1)^4\right) - ln(4x+1)^{\frac{1}{2}}$$

$$lny = \ln x^2 + ln(x+1)^4 - ln(4x+1)^{\frac{1}{2}}$$

$$lny = \ln x^2 + ln(x+1)^4 - ln(4x+1)^{\frac{1}{2}}$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 \cdot \frac{1}{x} + 4 \cdot \frac{1}{(x+1)} \cdot 1 - \frac{1}{2} \cdot \frac{1}{(4x+1)} \cdot 4$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{2}{x} + \frac{4}{(x+1)} - \frac{2}{(4x+1)}$$

$$\frac{dy}{dx} = y\left(\frac{2}{x} + \frac{4}{(x+1)} - \frac{2}{(4x+1)}\right)$$

$$\frac{dy}{dx} = \frac{x^2(x+1)^4}{\sqrt{4x+1}}\left(\frac{2}{x} + \frac{4}{(x+1)} - \frac{2}{(4x+1)}\right)$$

$$\frac{dy}{dx} = \frac{2x^2(x+1)^4}{\sqrt{4x+1}}\left(\frac{1}{x} + \frac{2}{(x+1)} - \frac{1}{(4x+1)}\right)$$

Example 4

Given $y = \frac{xe^{x^2}}{\sqrt{sinx}}$ use logarithmic differentiation to find $\frac{dy}{dx}$

$$y = \frac{xe^{x^2}}{\sqrt{sinx}}$$

$$lny = ln\left(\frac{xe^{x^2}}{\sqrt{sinx}}\right)$$

$$lny = ln\left(xe^{x^2}\right) - \ln\,(sinx)^{\frac{1}{2}}$$

$$lny = lnx + lne^{x^2} - \ln\,(sinx)^{\frac{1}{2}}$$

$$lny = lnx + x^2 - \tfrac{1}{2}\ln(sinx)$$

$$\ln y = \ln x + x^2 - \tfrac{1}{2}\ln(\sin x)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x} + 2x - \frac{1}{2} \cdot \frac{1}{\sin x} \cdot \cos x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x} + 2x - \tfrac{1}{2}\cot x$$

$$\frac{dy}{dx} = y\left(\frac{1}{x} + 2x - \tfrac{1}{2}\cot x\right)$$

$$\frac{dy}{dx} = \underline{\underline{\frac{xe^{x^2}}{\sqrt{\sin x}}\left(\frac{1}{x} + 2x - \tfrac{1}{2}\cot x\right)}}$$

Example 5

Given $y = x^x$ use logarithmic differentiation to find $\frac{dy}{dx}$

$$y = x^x$$

$$\ln y = \ln(x^x)$$

$$\ln y = x\ln x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x$$

$$\frac{dy}{dx} = y(1 + \ln x)$$

$$\frac{dy}{dx} = \underline{\underline{x^x(1 + \ln x)}}$$

Parametric Differentiation

Consider a point moving on the x, y plane.

Let (x, y) be the coordinates of the point at the time 't'.

Then both x and y are functions of t.

Suppose, for example, that $x = t^2$ and $y = 2t$.

When $t = 3$: $x = 3^2 = 9$ and $y = 2(3) = 6$.

Hence, the coordinates of the point at time $t = 3$ are (9, 6).

The equations for x and y, in terms of t, are known as **Parametric Equations** and 't' is known as the parameter.

An expression for $\frac{dy}{dx}$ in terms of t can be found by using the formula below:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

Example 1

A curve is defined by the parametric equations

$$x = t^2 + \frac{1}{t^2} \qquad y = t^2 - \frac{1}{t^2} \qquad , t \neq 0$$

Find an expression for $\frac{dy}{dx}$ in terms of t.

Give your answer in its simplest form.

Solution

$$x = t^2 + \frac{1}{t^2} \qquad y = t^2 - \frac{1}{t^2}$$

$$x = t^2 + t^{-2} \qquad y = t^2 - t^{-2}$$

$$\frac{dx}{dt} = 2t - 2t^{-3} \qquad \frac{dy}{dt} = 2t + 2t^{-3}$$

$$\frac{dx}{dt} = 2t - \frac{2}{t^3} \qquad \frac{dy}{dt} = 2t + \frac{2}{t^3}$$

$$\frac{dx}{dt} = 2t - \frac{2}{t^3} \qquad \frac{dy}{dt} = 2t + \frac{2}{t^3}$$

$$\frac{dy}{dx} = \frac{2t + \frac{2}{t^3}}{2t - \frac{2}{t^3}}$$

$$\frac{dy}{dx} = \frac{\left(2t + \frac{2}{t^3}\right)}{\left(2t - \frac{2}{t^3}\right)} \times \frac{t^3}{t^3}$$

$$\frac{dy}{dx} = \frac{2t^4 + 2}{2t^4 - 2}$$

$$\frac{dy}{dx} = \frac{t^4 + 1}{t^4 - 1}$$

Example 2

A curve is defined by the parametric equations $x = \theta - sin\theta \quad y = 1 - cos\theta$

$(0 \le \theta \le 2\pi)$

Find an expression for $\frac{dy}{dx}$ in terms of θ

Solution

$$x = \theta - sin\theta \qquad y = 1 - cos\theta$$

$$\frac{dx}{dt} = 1 - cos\theta \qquad \frac{dy}{dt} = sin\theta$$

$$\frac{dy}{dx} = \frac{sin\theta}{1 - cos\theta}$$

Example 3

A curve is defined by the parametric equations $x = \frac{t}{1+t}$ $y = \frac{1+t}{1-t}$ $(t \neq \pm 1)$

Solution

$x = \frac{t}{1+t}$

$u = t \quad u' = 1$

$v = 1+t \quad v' = 1$

$\frac{dx}{dt} = \frac{u'v - v'u}{v^2}$

$\frac{dx}{dt} = \frac{1(1+t) - 1(t)}{(1+t)^2}$

$\frac{dx}{dt} = \frac{1}{(1+t)^2}$

$y = \frac{1+t}{1-t}$

$u = 1+t \quad u' = 1$

$v = 1-t \quad v' = -1$

$\frac{dy}{dt} = \frac{u'v - v'u}{v^2}$

$\frac{dy}{dt} = \frac{1(1-t) - (-1)(1+t)}{(1-t)^2}$

$\frac{dy}{dt} = \frac{2}{(1-t)^2}$

$\frac{dx}{dt} = \frac{1}{(1+t)^2}$

$\frac{dy}{dt} = \frac{2}{(1-t)^2}$

$$\frac{dy}{dx} = \frac{\frac{2}{(1-t)^2}}{\frac{1}{(1+t)^2}}$$

$$\frac{dy}{dx} = \frac{2(1+t)^2}{(1-t)^2}$$

Example 4

A curve is defined by the parametric equations $x = 1 - t^2 \quad y = t^3 + t$

at which the gradient is 2.

Solution

$$x = 1 - t^2 \qquad y = t^3 + t$$

$$\frac{dx}{dt} = -2t \qquad \frac{dy}{dt} = 3t^2 + 1$$

$$\frac{dy}{dx} = \frac{3t^2+1}{-2t}$$

When $m = 2$

$$\frac{3t^2+1}{-2t} = 2$$

$$3t^2 + 1 = -4t$$

$$3t^2 + 4t + 1 = 0$$

$$(3t+1)(t+1) = 0 \qquad \therefore t = -\tfrac{1}{3}, \ -1$$

When $t = -1$

$$x = 1 - t^2 \qquad y = t^3 + t$$

$$x = 1 - (-1)^2 \qquad y = (-1)^3 + (-1)$$

$$= 0 \qquad = -2$$

i.e. (0, -2)

When $t = -\frac{1}{3}$

$$x = 1 - t^2 \qquad y = t^3 + t$$

$$x = 1 - \left(-\tfrac{1}{3}\right)^2 \qquad y = \left(-\tfrac{1}{3}\right)^3 + \left(-\tfrac{1}{3}\right)$$

$$x = \frac{8}{9} \qquad y = -\frac{10}{27}$$

i.e. $\left(\frac{8}{9}, -\frac{10}{27}\right)$

Differentiation of Inverse Trigonometric Functions

$f(x)$	$f'(x)$
$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1}x$	$\frac{1}{1+x^2}$

Example 1

$$y = \sin^{-1}(5x)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(5x)^2}} \cdot 5$$

$$\frac{dy}{dx} = \frac{5}{\sqrt{1-25x^2}}$$

Example 2

$$y = cos^{-1}(x^2)$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^4}} \;.\; 2x$$

$$\frac{dy}{dx} = -\frac{2x}{\sqrt{1-x^4}}$$

Example 3

$$y = tan^{-1}(\sqrt{3x-1})$$

$$y = tan^{-1}\left((3x-1)^{\frac{1}{2}}\right)$$

$$\frac{dy}{dx} = \frac{1}{1+\left((3x-1)^{\frac{1}{2}}\right)^2} \;.\; \frac{1}{2}(3x-1)^{-\frac{1}{2}} \;.\; 3$$

$$\frac{dy}{dx} = \frac{1}{1+(3x-1)} \;.\; \frac{3}{2(3x-1)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{1}{3x} \;.\; \frac{3}{2\sqrt{3x-1}}$$

$$\frac{dy}{dx} = \frac{3}{6x\sqrt{3x-1}}$$

$$\frac{dy}{dx} = \frac{1}{2x\sqrt{3x-1}}$$

Example 4

$$y = \sqrt{1 - x^2}\, cos^{-1}(x)$$

$$u = \left(1 - x^2\right)^{\frac{1}{2}} \qquad v = cos^{-1}(x)$$

$$u' = \tfrac{1}{2}\left(1 - x^2\right)^{-\frac{1}{2}} \;.\; (-2x) \qquad v' = -\frac{1}{\sqrt{1-x^2}}$$

$$u' = \frac{(-2x)}{2\sqrt{1-x^2}}$$

$$u' = -\frac{x}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = u'v + v'u$$

$$\frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}} \;.\; cos^{-1}(x) + \left(-\frac{1}{\sqrt{1-x^2}}\right) . \sqrt{1 - x^2}$$

$$\frac{dy}{dx} = -\frac{x cos^{-1}(x)}{\sqrt{1-x^2}} - 1$$

Past Paper Example

(a) Show that $\frac{d}{dx}\left(\frac{e^x}{x}\right) = \frac{(x-1)e^x}{x^2}$

(b) Hence given $y = tan^{-1}\left(\frac{e^x}{x}\right)$, show that $\frac{dy}{dx} = \frac{(x-1)e^x}{x^2+e^{2x}}$

(a) $u = e^x \qquad u' = e^x$

$v = x \qquad v' = 1$

$$\frac{dy}{dx} = \frac{u'v - v'u}{v^2}$$

$$\frac{dy}{dx} = \frac{e^x.x - e^x}{x^2}$$

$$\frac{dy}{dx} = \frac{e^x(x-1)}{x^2}$$ *as required*

$f(x) = tan^{-1} \qquad f'(x) = \frac{1}{1+x^2}$

$$y = tan^{-1}\left(\frac{e^x}{x}\right)$$

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{e^x}{x}\right)^2} \cdot \frac{e^x(x-1)}{x^2}$$

$$\frac{dy}{dx} = \frac{1}{1+\frac{e^{2x}}{x^2}} \cdot \frac{e^x(x-1)}{x^2}$$

$$\frac{dy}{dx} = \frac{e^x(x-1)}{x^2\left(1+\frac{e^{2x}}{x^2}\right)}$$

$$\frac{dy}{dx} = \frac{e^x(x-1)}{x^2+e^{2x}}$$ *as required*