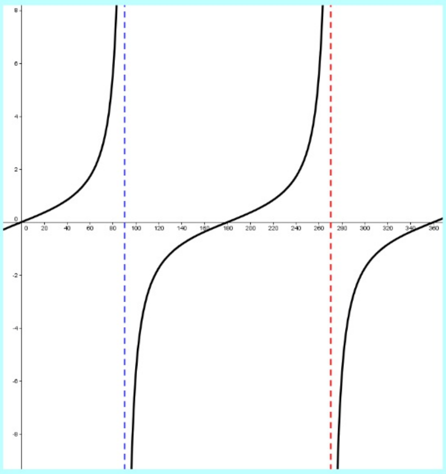
Curve Sketching

Asymptotes

An asymptote is a straight line to which a curve approaches more and more closely as x becomes larger or smaller, or approaches a certain value.

Example

At $x = 90^o$ the value of $\tan x$ is undefined $\therefore$ the line $x = 90$ is called an asymptote.



Vertical Asymptotes

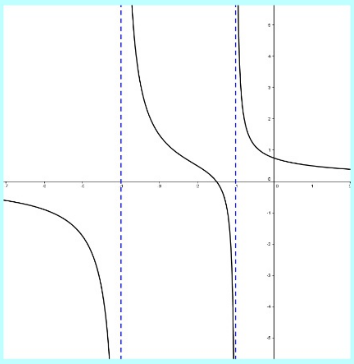
Vertical Asymptotes are found from the zeroes of the denominator and are in the form $x = k$.

The way the curve approaches the asymptote must also be determined.

Example

$$f(x) = \frac{2x+3}{x^2+5x+4}$$

$$f(x) = \frac{2x+3}{(x+1)(x+4)}$$



For a vertical asymptote (VA) the denominator is equal to zero.

So $(x + 4)(x + 1) = 0$

The eq^{n}s of the VA are $x = -4$ or $x = -1$

How does f(x) behave in the neighbourhood of x = -4 and x = -1?

Behaviour

$$f(x) = \frac{2x+3}{(x+1)(x+4)}$$

x = -4

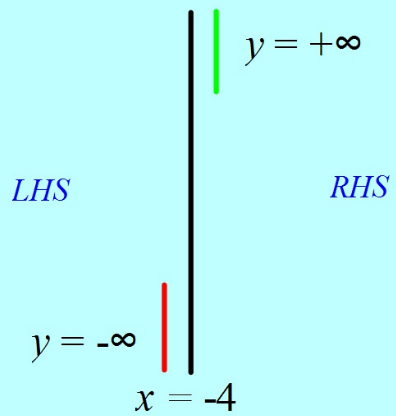
approaching from the left

the resulting 'signs' from subst. a number really close to the VA

As $x \to -4^{-}$ $\quad f(x) = \frac{(-)}{(-)(-)} \quad \to -\infty$

As $x \to -4^{+}$ $\quad f(x) = \frac{(-)}{(-)(+)} \quad \to +\infty$

and from the right

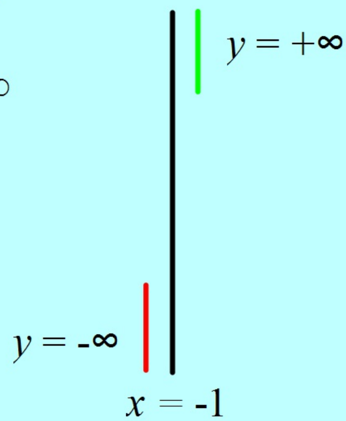


$$f(x) = \frac{2x+3}{(x+1)(x+4)}$$

x = -1

As $x \to -1^{-}$ $\quad f(x) = \frac{(+)}{(-)(+)} \quad \to -\infty$

As $x \to -1^{+}$ $\quad f(x) = \frac{(+)}{(+)(+)} \quad \to +\infty$



Non Vertical Asymptotes

Non Vertical Asymptotes are in the form y = c (horizontal) or $y = mx + c$ (slant or oblique).

The way the curve approaches the asymptote must also be determined.

We can find NVA by considering how the function behaves as x→∞.

Top Tips

1) if the degree of the numerator ≥ degree of the denominator, divide the numerator by the denomator using long division

2) if the degree of the numerator < degree of the denominator, divide each term by the highest power of x.

Example 1

Find the NVA for the function $f(x) = \frac{2x+3}{x^2+5x+4}$

$$f(x) = \frac{2x+3}{x^2+5x+4}$$

deg top < deg bottom; divide each term by highest power of x

$$= \frac{\frac{2x}{x^2} + \frac{3}{x^2}}{\frac{x^2}{x^2} + \frac{5x}{x^2} + \frac{4}{x^2}}$$

$$= \frac{\frac{2}{x} + \frac{3}{x^2}}{1 + \frac{5}{x} + \frac{4}{x^2}}$$

As $x \to \pm\infty$ (large positive and negative values of x) all tend to zero.

Hence NVA eq^n is y = 0 *(a horizontal asymptote)*

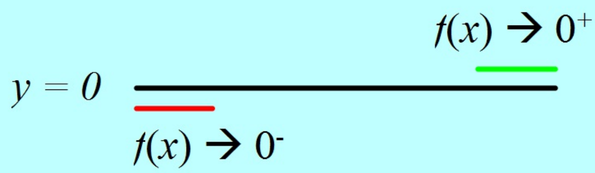
Behaviour

$$f(x) = \frac{\frac{2}{x} + \frac{3}{x^2}}{1 + \frac{5}{x} + \frac{4}{x^2}}$$

Let us now consider how the curve approaches $y = 0$.

As $x \to +\infty$ $\quad f(x) \to 0^+$ *(above the value of 0, since the fraction will be very small and positive)*

As $x \to -\infty$ $\quad f(x) \to 0^-$ *(below the value of 0, since the fraction will be very small and negative)*



Example 2

$$f(x) = \frac{x^2 + 2x + 1}{x^2 + 5x + 4}$$ *deg top = deg bottom; use long division*

$$\begin{array}{rrrrrrr}
 & & & & & & 1 \\
x^2 + 5x + 4 & \big) & x^2 & + & 2x & + & 1 \\
 & - & x^2 & + & 5x & + & 4 \\
\hline
 & & & - & 3x & - & 3
\end{array}$$

$$Q + \frac{r}{d}$$

$$f(x) = 1 - \frac{3x + 3}{x^2 + 5x + 4}$$

$$f(x) = 1 - \frac{3x+3}{x^2+5x+4}$$

deg top < deg bottom; divide each term by highest power of x

$$f(x) = 1 - \frac{\frac{3x}{x^2} + \frac{3}{x^2}}{\frac{x^2}{x^2} + \frac{5x}{x^2} + \frac{4}{x^2}}$$

$$f(x) = 1 - \frac{\frac{3}{x} + \frac{3}{x^2}}{1 + \frac{5}{x} + \frac{4}{x^2}}$$

As $x \to \pm\infty$ (large positive and negative values of x) all tend to zero.

Hence NVA eq^n is $f(x) = 1$ *(a horizontal asymptote)*.

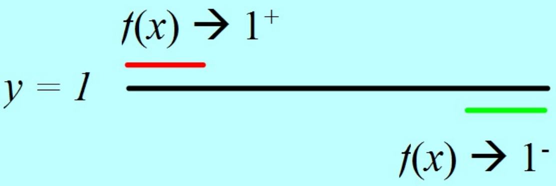
Behaviour

$$f(x) = 1 - \frac{\frac{3}{x} + \frac{3}{x^2}}{1 + \frac{5}{x} + \frac{4}{x^2}}$$

Let us consider how the curve approaches $y = 1$.

As $x \to +\infty$ $f(x) \to 1^-$ *(below the value of 1, since the fraction will be very small and positive)*

As $x \to -\infty$ $f(x) \to 1^+$ *(above the value of 1, since the fraction is negative)*



Example 3

$$f(x) = \frac{x^2+4x+3}{x+2}$$

deg top > deg bottom; use long division

$$\begin{array}{r|rrrrrrr} & & & x & + & 2 \\ \cline{2-6} x+2 & x^2 & + & 4x & + & 3 \\ - & x^2 & + & 2x & & \downarrow \\ \cline{2-6} & & & 2x & + & 3 \\ & & - & 2x & + & 4 \\ \cline{4-6} & & & & - & 1 \end{array}$$

$$Q + \frac{r}{d}$$

$$f(x) = x + 2 - \frac{1}{x+2}$$

$$f(x) = x + 2 - \frac{1}{x+2}$$

deg top < deg bottom; divide each term by highest power of x

$$f(x) = x + 2 - \frac{\frac{1}{x}}{\frac{x}{x}+\frac{2}{x}}$$

$$f(x) = x + 2 - \frac{\frac{1}{x}}{1+\frac{2}{x}}$$

As $x \to \pm\infty$ (large positive and negative values of x) all tend to 0.

Hence NVA eq^n is $f(x) = x + 2$ *(an oblique asymptote)*.

Behaviour

$$f(x) = x + 2 - \frac{\frac{1}{x}}{1+\frac{2}{x}}$$

Let us consider how the curve approaches $y = x + 2$.

As $x \to +\infty$ $f(x) \to (x+2)^-$

As $x \to -\infty$ $f(x) \to (x+2)^+$