

Differential Equations - First Order

2012

Q5b – 7 marks [part (a) worth 4]

(a) Express $\frac{1}{(x-1)(x+2)^2}$ in partial fractions.

Marks
4

(b) Obtain the general solution of the differential equation

$$(x-1)\frac{dy}{dx} - y = \frac{x-1}{(x+2)^2},$$

expressing your answer in the form $y = f(x)$.

7

Marking Instructions

(a) $\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$ 1M

$$1 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$$

$$x = 1 \Rightarrow A = \frac{1}{9} \quad 1$$

$$x = -2 \Rightarrow C = -\frac{1}{3} \quad 1$$

$$x = 0 \Rightarrow 1 = \frac{4}{9} - 2B + \frac{1}{3} \Rightarrow B = -\frac{1}{9} \quad 1$$

$$\therefore \frac{1}{(x-1)(x+2)^2} = \frac{1}{9} \left(\frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2} \right)$$

(b) $(x-1)\frac{dy}{dx} - y = \frac{x-1}{(x+2)^2}$

$$\frac{dy}{dx} - \frac{1}{x-1}y = \frac{1}{(x+2)^2} \quad 1M \text{ for rearranging}$$

Integrating factor: $\exp\left(\int -\frac{1}{x-1}dx\right)$ 1

$$= \exp(-\ln(x-1)) = (x-1)^{-1} \quad 1$$

$$\frac{1}{(x-1)}\frac{dy}{dx} - \frac{1}{(x-1)^2}y = \frac{1}{(x-1)(x+2)^2}$$

$$\frac{d}{dx}\left(\frac{y}{x-1}\right) = \frac{1}{(x-1)(x+2)^2} \quad 1$$

$$= \frac{1}{9}\left(\frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2}\right) \quad 1$$

$$\frac{y}{x-1} = \frac{1}{9}\left(\ln|x-1| - \ln|x+2| + \frac{3}{x+2}\right) + c \quad 1 \text{ constant of integration needed.}$$

$$y = \frac{x-1}{9}\left(\ln|x-1| - \ln|x+2| + \frac{3}{x+2}\right) + c(x-1) \quad 1$$

$$= \frac{x-1}{9}\left(\ln\left|\frac{x-1}{x+2}\right| + \frac{3}{x+2}\right) + c(x-1)$$

2011

Q9 – 5 marks

Given that $y > -1$ and $x > -1$, obtain the general solution of the differential equation

$$\frac{dy}{dx} = 3(1+y)\sqrt{1+x}$$

expressing your answer in the form $y = f(x)$.

5

Marking Instructions

Method 1

$$\frac{dy}{dx} = 3(1+y)\sqrt{1+x}$$

$$\int \frac{dy}{1+y} = 3 \int (1+x)^{\frac{1}{2}} dx$$

$$\ln(1+y) = 2(1+x)^{\frac{3}{2}} + c$$

$$1+y = \exp(2(1+x)^{\frac{3}{2}} + c)$$

$$y = \exp(2(1+x)^{\frac{3}{2}} + c) - 1.$$

$$= A \exp(2(1+x)^{\frac{3}{2}}) - 1.$$

M1 separating variables

1 for LHS

1 for term in x

1 for the constant

1

2009

Q3 – 4 marks

Given that

$$x^2 e^y \frac{dy}{dx} = 1$$

and $y = 0$ when $x = 1$, find y in terms of x .

4

Marking Instructions

$$e^y x^2 \frac{dy}{dx} = 1$$

$$e^y \frac{dy}{dx} = x^{-2}$$

$$\int e^y dy = \int x^{-2} dx \quad \mathbf{1}$$

$$e^y = -x^{-1} + c \quad \mathbf{1}$$

$y = 0$ when $x = 1$ so

$$1 = -1 + c \Rightarrow c = 2 \quad \mathbf{1}$$

$$e^y = 2 - \frac{1}{x} \Rightarrow y = \ln\left(2 - \frac{1}{x}\right) \quad \mathbf{1}$$

2013

Q16 – 10 marks

In an environment without enough resources to support a population greater than 1000, the population $P(t)$ at time t is governed by Verhurst's law

$$\frac{dP}{dt} = P(1000 - P).$$

Show that

$$\ln \frac{P}{1000 - P} = 1000t + C \quad \text{for some constant } C. \quad 4$$

Hence show that

$$P(t) = \frac{1000K}{K + e^{-1000t}} \quad \text{for some constant } K. \quad 3$$

Given that $P(0) = 200$, determine at what time t , $P(t) = 900$. 3

Marking Instructions

$\frac{dP}{dt} = P(1000 - P)$		
$\text{So } \int \frac{dP}{P(1000 - P)} = \int dt$		<ul style="list-style-type: none"> •¹ Separates variables.⁵
$\frac{1}{P(1000 - P)} = \frac{A}{P} + \frac{B}{1000 - P}$		<ul style="list-style-type: none"> •² Appropriate form of partial fractions.
$A = \frac{1}{1000}, B = \frac{1}{1000}$		<ul style="list-style-type: none"> •³ Obtains correct values of both A and B.
$\frac{1}{1000} \int \left(\frac{1}{P} + \frac{1}{1000 - P} \right) dP = \int dt$		
$\ln P - \ln(1000 - P) = 1000t + c$		<ul style="list-style-type: none"> •⁴ Integrates correctly, including '+c'.⁶
$\ln \frac{P}{1000 - P} = 1000t + c$		
$\frac{P}{1000 - P} = Ke^{1000t} \text{ (where } K = e^c \text{)}$		<ul style="list-style-type: none"> •⁵ Accurately converts to exponential form.¹

$$P = 1000Ke^{1000t} - PKe^{1000t},$$

$$P + PKe^{1000t} = 1000Ke^{1000t},$$

$$P = \frac{1000Ke^{1000t}}{1 + Ke^{1000t}}$$

$$= \frac{1000K}{e^{-1000t} + K} \quad \left(\text{or } \frac{1000e^c}{e^{-1000t} + e^c} \right)$$

- ⁶ Multiplies out fractions and collects P terms.

- ⁷ Factorises and divides to obtain required form.²

$$\text{Since } P(0) = 200, \quad 200 = \frac{1000K}{1 + K}$$

$$K = \frac{1}{4} \quad (\text{or } 0.25)$$

$$\text{Require } 900 = \frac{1000 \times 0.25}{0.25 + e^{-1000t}}$$

$$225 + 900e^{-1000t} = 250$$

$$e^{1000t} = 36$$

$$1000t = \ln 36$$

$$t = \frac{1}{1000} \ln 36$$

$$[\text{or } 0.003584 \text{ (4sf)}]$$

- ⁸ Equates and process to obtain value of K .³

- ⁹ Inserts value of K and equates.

- ¹⁰ Solves to obtain value for t .⁴

2007

Q14 – 10 marks

A garden centre advertises young plants to be used as hedging.

After planting, the growth G metres (ie the increase in height) after t years is modelled by the differential equation

$$\frac{dG}{dt} = \frac{25k - G}{25}$$

where k is a constant and $G = 0$ when $t = 0$.

- | | | |
|-----|--|---|
| (a) | Express G in terms of t and k . | 4 |
| (b) | Given that a plant grows 0.6 metres by the end of 5 years, find the value of k correct to 3 decimal places. | 2 |
| (c) | On the plant labels it states that the expected growth after 10 years is approximately 1 metre. Is this claim justified? | 2 |
| (d) | Given that the initial height of the plants was 0.3 m, what is the likely long-term height of the plants? | 2 |

Marking Instructions

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|-----|---|---|
| (a) | $\frac{dG}{dt} = \frac{25k - G}{25}$ | |
| | $\int \frac{dG}{25k - G} = \int \frac{1}{25} dt$ | 1 |
| | $-\ln(25k - G) = \frac{t}{25} + C$ | 1 |
| | When $t = 0, G = 0$, so $C = -\ln 25k$ | 1 |
| | $25k - G = 25ke^{-t/25}$ | |
| | $G = 25k(1 - e^{-t/25})$ | 1 |
| (b) | When $t = 5, G = 0.6$. Therefore | |
| | $0.6 = 25k(1 - e^{-0.2})$ | 1 |
| | $k = 0.6 / (25(1 - e^{-0.2})) \approx 0.132$ | 1 |
| (c) | When $t = 10$ | |
| | $G \approx 3.3(1 - e^{-0.4})$ | 1 |
| | ≈ 1.09 | |
| | The claim seems to be justified, | 1 |
| (d) | As $t \rightarrow \infty, G \rightarrow 25k \approx 3.3$ metres | 1 |
| | so the limit is 3.6 metres. | 1 |