Further Integration

2013

Q8 – 5 marks

Use integration by parts to obtain $\int x^2 \cos 3x \, dx$.

Written Solutions

$$\int \underset{d}{x^2} \, \underset{i}{\cos 3x}$$

$$= x^2 \cdot \tfrac{1}{3}\sin 3x - \int 2x \cdot \tfrac{1}{3}\sin 3x \, dx$$

$$= \tfrac{1}{3}x^2 \sin 3x - \tfrac{2}{3}\int \underset{d}{x} \, \underset{i}{\sin 3x} \, dx$$

$$= \tfrac{1}{3}x^2 \sin 3x - \tfrac{2}{3}\left[x\left(-\tfrac{1}{3}\right)\cos 3x - \int \left(-\tfrac{1}{3}\right)\cos 3x \, dx\right]$$

$$= \tfrac{1}{3}x^2 \sin 3x + \tfrac{2}{9}x\cos 3x - \tfrac{2}{9}\int \cos 3x \, dx$$

$$= \tfrac{1}{3}x^2 \sin 3x + \tfrac{2}{9}x\cos 3x - \tfrac{2}{27}\sin 3x + c$$

$$= \underline{\left(\tfrac{1}{3}x^2 - \tfrac{2}{27}\right)\sin 3x + \tfrac{2}{9}x\cos 3x + c}$$

2012

Q11b – 4 marks

i) Write down the derivative of $\sin^{-1}x$.

ii) Use integration by parts to obtain $\int \sin^{-1}x \cdot \frac{x}{\sqrt{1-x^2}}\,dx$.

Written Solutions

a) $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

b) $\int \sin^{-1}x \cdot \frac{x}{\sqrt{1-x^2}}\,dx$

d i

$= \sin^{-1}x\left(-\sqrt{1-x^2}\right) - \int \frac{1}{\sqrt{1-x^2}} \cdot \left(-\sqrt{1-x^2}\right)dx$

$= -\sqrt{1-x^2}\sin^{-1}x + \int dx$

$= -\sqrt{1-x^2}\sin^{-1}x + x + C$

$\int \frac{x}{\sqrt{1-x^2}}\,dx$ let $x = \sin\theta$, $\frac{dx}{d\theta} = \cos\theta$

$= \int \sin\theta\,d\theta$

$= -\cos\theta$

$= -\sqrt{1-x^2}$

2011

Q16a – 3 marks

Define $I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,dx$ for $n \geq 1$.

(*a*) Use integration by parts to show that

$$I_n = \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx.$$

Written Solutions

$$I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,dx$$

$$= \int_0^1 \underset{i}{1} \cdot \underset{d}{\frac{1}{(1+x^2)^n}}\,dx$$

$$= \left[x \cdot \frac{1}{(1+x^2)^n}\right]_0^1 - \int_0^1 x \cdot (-n) \cdot \frac{2x}{(1+x^2)^{n+1}}\,dx$$

$$= \left(1 \cdot \frac{1}{(1+1)^n} - 0\right) + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx$$

$$= \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx$$

2010

Q3b – 4 marks

Integrate $x^2 \ln x$ with respect to x. **4**

Written Solutions

$$\int x^2 \ln x \, dx$$

$$= \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \cdot \frac{1}{x}\, dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2\, dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$$

$$= \frac{1}{9}x^3(3\ln x - 1) + C$$

2009

Q9 – 5 marks

Use integration by parts to obtain the exact value of $\int_0^1 x \tan^{-1} x^2 \, dx$.

Written Solutions

$$\int_0^1 \underset{i}{x} \underset{d}{\tan^{-1} x^2} \, dx$$

$$= \left[\frac{1}{2}x^2 \tan^{-1} x^2\right]_0^1 - \int_0^1 \frac{1}{2}x^2 \cdot \frac{1}{1+x^4} \cdot 2x \, dx$$

$$= \left[\frac{1}{2}x^2 \tan^{-1} x^2\right]_0^1 - \int_0^1 \frac{x^3}{1+x^4} \, dx$$

$$= \left[\frac{1}{2}x^2 \tan^{-1} x^2\right]_0^1 - \left[\frac{1}{4}\ln(1+x^4)\right]_0^1$$

$$= \underline{\underline{\frac{\pi}{8} - \frac{1}{4}\ln 2}}$$

2008

Q7 – 5 marks

Use integration by parts to obtain $\int 8x^2 \sin 4x \, dx$.

Written Solutions

$$\int \underset{d}{8x^2}\, \underset{i}{\sin 4x}\, dx$$

$$= 8x^2 \cdot \left(-\tfrac{1}{4}\cos 4x\right) - \int \underset{d}{16x}\left(-\underset{i}{\tfrac{1}{4}\cos 4x}\right) dx$$

$$= -2x^2 \cos 4x + 4 \int x \cos 4x \, dx$$

$$= -2x^2 \cos 4x + 4\left(x \cdot \tfrac{1}{4}\sin 4x - \int 1 \cdot \tfrac{1}{4} \sin 4x \, dx \right)$$

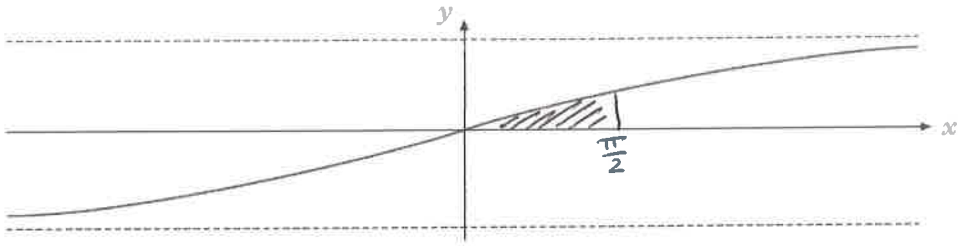
$$= -2x^2 \cos 4x + x \sin 4x - \int \sin 4x \, dx$$

$$= -2x^2 \cos 4x + x \sin 4x + \tfrac{1}{4}\cos 4x + c$$

$$= \underline{\underline{x \sin 4x + \left(\tfrac{1}{4} - 2x^2\right)\cos 4x + c}}$$

2007

Q16b – 5 marks



(*a*) The diagram shows part of the graph of $f(x) = \tan^{-1} 2x$ and its asymptotes. State the equations of these asymptotes. **2**

(*b*) Use integration by parts to find the area between $f(x)$, the x-axis and the lines $x = 0$, $x = \frac{1}{2}$. **5**

(*c*) Sketch the graph of $y = |f(x)|$ and calculate the area between this graph, the x-axis and the lines $x = -\frac{1}{2}$, $x = \frac{1}{2}$. **3**

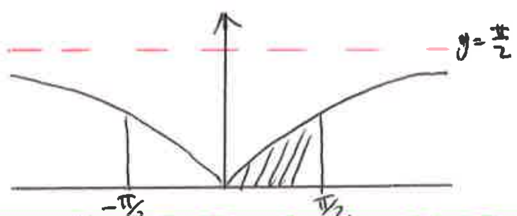
Written Solutions

(a) $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$

(b)
$$A = \int_0^{\frac{1}{2}} \tan^{-1}(2x)\,dx$$
$$= \int_0^{\frac{1}{2}} \underset{i}{1} \cdot \underset{d}{\tan^{-1}(2x)}\,dx$$
$$= \Big[x \tan^{-1}(2x)\Big]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} x \cdot \frac{1}{1+(2x)^2} \cdot 2\,dx$$
$$= \left(\tfrac{1}{2}\tan^{-1}(1) - 0\right) - \int_0^{\frac{1}{2}} \frac{2x}{1+4x^2}\,dx$$
$$= \frac{1}{2} \cdot \frac{\pi}{4} - \frac{1}{4}\int_0^{\frac{1}{2}} \frac{8x}{1+4x^2}\,dx$$
$$= \frac{\pi}{8} - \frac{1}{4}\Big[\ln(1+4x^2)\Big]_0^{\frac{1}{2}}$$
$$= \frac{\pi}{8} - \frac{1}{4}\ln 2$$

numerator is a multiple of the derivative of the denominator / we have a multiple of the logarithm of the denominator.

(c)



$$A = 2\left(\frac{\pi}{8} - \frac{1}{4}\ln 2\right)$$
$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$