SECOND ORDER

Differential Equations

Homogeneous Second Order Differential Equations

The equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

This is called a homogeneous equation because the right hand side is zero.

The general solution of this equation depends on the quadratic equation $am^2 + bm + c = 0$.

This is called the auxiliary equation. There are three separate cases to consider, depending on the nature of the roots.

These three cases are:

- real unequal roots
- equal roots
- complex roots

General solutions for these three cases are shown below. Note that *A* and *B* are arbitrary constants.

Real unequal roots m_1, m_2. The general solution is:

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

General solutions for these three cases are shown below. Note that *A* and *B* are arbitrary constants.

Equal roots m_1, m_2. The general solution is:

$$y = (Ax + B)e^{mx}$$

General solutions for these three cases are shown below. Note that *A* and *B* are arbitrary constants.

Complex roots $p \pm qi$. The general solution is:

$$y = e^{px}(A\sin qx + B\cos qx)$$

Ex 1. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

A.E $\Rightarrow$ $m^2 + m - 6 = 0$ $\quad (b^2 - 4ac = 25)$

$(m + 3)(m - 2) = 0$ $\quad \Rightarrow m_1 = -3,\ m_2 = 2$

The roots are real and distinct

G.S $\quad y = Ae^{m_1x} + Be^{m_2x} \quad \Rightarrow \quad y = Ae^{-3x} + Be^{2x}$

Ex 2. Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

A.E $\Rightarrow$ $m^2 + 4m + 4 = 0$ $\quad (b^2 - 4ac = 0)$

$(m + 2)^2 = 0$ $\quad \Rightarrow m = -2$ (double root)

The roots are real and equal

G.S $\quad y = (Ax + B)e^{mx} \quad \Rightarrow \quad y = (Ax + B)\, e^{-2x}$

Ex 3. Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$

A.E $\Rightarrow$ $m^2 + 2m + 5 = 0$ doesn't factorise.

$a = 1, b = 2, c = 5$ $(b^2 - 4ac = -16)$

$$m = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2}$$

$$= -1 \pm 2i$$

G.S $y = e^{px}\left(A\sin qx + B\cos qx\right)$

$\Rightarrow$ $y = e^{-x}\left(A\sin 2x + B\cos 2x\right)$

Homogeneous Second Order Differential Equations Worksheet Q1

So far, we have been finding the ***General Solution*** of second-order differential equations, leaving our answer with two arbitrary constants *A* and *B*.

The next step is to fully solve or find the ***Particular Solution***.

To evaluate two aritrary constants, we need two pieces of information which come in the form of initial conditions, such as where the values of *y* and $\frac{dy}{dx}$ are given for a value of *x*.

Ex 4. Find the particular solution of $2\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 4y = 0$

Given that when $x = 0$, $y = 1$ and $\frac{dy}{dx} = 2$

A.E $\Rightarrow$ $2m^2 + 7m - 4 = 0$

$(2m - 1)(m + 4) = 0$ $\Rightarrow m_1 = ½ , m_2 = -4$

G.S $y = Ae^{\frac{1}{2}x} + Be^{-4x}$

Using initial conditions, $x = 0$ and $y = 1$

$1 = Ae^0 + Be^0$ $\Rightarrow 1 = A + B$

Using $x = 0$ and $\frac{dy}{dx} = 2$

$$\frac{dy}{dx} = \frac{1}{2}Ae^{\frac{1}{2}x} - 4Be^{-4x}$$

$2 = \frac{1}{2}Ae^0 - 4Be^0$ $\Rightarrow 2 = ½A - 4B$

Solving these equations gives $A = \frac{4}{3}, B = -\frac{1}{3}$

Particular Solution $y = \frac{4}{3}e^{\frac{1}{2}x} - \frac{1}{3}e^{-4x}$

Ex 5. Find the particular solution to the equation $9y'' - 6y' + y = 0$ given $y = 1, x = 0$ and $y = 4e, x = 3$.

A.E $\Rightarrow 9m^2 - 6m + 1 = 0$

$(3m - 1)^2 = 0 \qquad \Rightarrow m = \frac{1}{3}$

G.S $y = (A + Bx)\, e^{\frac{1}{3}x}$

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Using $y = 1$ and $x = 0$

$1 = (A + B(0))e^0 \qquad \Rightarrow 1 = A$

Using $y = 4e$ and $x = 3$

$4e = (A + B(3))e^1 \qquad \Rightarrow 4e = Ae + 3Be$

$4 = A + 3B \quad \Rightarrow B = 1$

Particular solution $\quad y = (1 + x)e^{\frac{1}{3}x}$

Ex 6. Find the particular solution to $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$,

given that when $x = 0, y = 2$ and $\frac{dy}{dx} = 0$

A.E $\Rightarrow m^2 + 4m + 13 = 0 \qquad$ doesn't factorise $\qquad b^2 - 4ac = -36$

$m = \frac{-4 \pm \sqrt{16 - 52}}{2} = \frac{-4 \pm \sqrt{-36}}{2}$

$= -2 \pm 3i$

G.S $y = e^{-2x}\left(A\sin 3x + B\cos 3x\right)$

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Using, $x = 0$ and $y = 2$

$\Rightarrow 2 = e^0\,(A \sin 0 + B \cos 0) \quad \Rightarrow 2 = B$

Using, $x = 0$ and $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -2e^{-2x}\left(A\sin 3x + B\cos 3x\right) + e^{-2x}\left(3A\cos 3x - 3B\sin 3x\right)$

$\Rightarrow 0 = -2e^0\,(B) + e^0(3A)$

$0 = -2B + 3A \qquad \Rightarrow A = \frac{4}{3}$

Particular solution $y = e^{-2x}\,(\frac{4}{3}\sin 3x + 2\cos 3x)$

2010

Q11 – 7 marks

Obtain the general solution of the equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0.$$

Hence obtain the solution for which $y = 3$ when $x = 0$ and $y = e^{-\pi}$ when $x = \frac{\pi}{2}$.

2014

Q8 – 6 marks

Find the solution $y = f(x)$ to the differential equation

$$4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$$

given that $y = 4$ and $\frac{dy}{dx} = 3$ when $x = 0$.

Homogeneous Particular Solutions Worksheet