Integration

2013

Q6 – 4 marks

Integrate $\dfrac{\sec^2 3x}{1+\tan 3x}$ with respect to x. **4**

Written Solutions

$$\int \frac{\sec^2 3x}{1+\tan 3x}\,dx$$

$$= \frac{1}{3}\int \frac{3\sec^2 3x}{1+\tan 3x}\,dx$$

$\therefore \frac{d}{dx}(1+\tan 3x) = 3\sec^2 3x$

$$= \frac{1}{3}\ln|1+\tan 3x| + c$$

OR

$$\int \frac{\sec^2 3x}{1+\tan 3x}\,dx$$

let $u = 1+\tan 3x$

$\frac{dy}{dx} = \sec^2 3x \cdot 3$

$dx = \frac{du}{3\sec^2 3x}$

$$= \int \frac{\sec^2 3x}{u} \cdot \frac{du}{3\sec^2 3x}$$

$$= \frac{1}{3}\int \frac{du}{u}$$

$$= \frac{1}{3}\ln|u| + c$$

$$= \frac{1}{3}\ln|1+\tan 3x| + c$$

2012

Q8 – 6 marks

Use the substitution $x = 4\sin\theta$ to evaluate $\int_0^2 \sqrt{16-x^2}\,dx$. 6

Written Solutions

$$\int_0^2 \sqrt{16-x^2}\,dx$$

let $x = 4\sin\theta$

$\frac{dx}{d\theta} = 4\cos\theta \Rightarrow dx = 4\cos\theta\,d\theta$

$x=0$: $4\sin\theta = 0$, $\theta = 0$

$x=2$: $4\sin\theta = 2$, $\sin\theta = \frac{1}{2}$, $\theta = \pi/6$

$$= \int_0^{\frac{\pi}{6}} \sqrt{16-16\sin^2\theta}\cdot 4\cos\theta\,d\theta$$

$\therefore$ $16 - 16\sin^2\theta = 16(1-\sin^2\theta) = 16\cos^2\theta$

$$= \int_0^{\pi/6} 4\cos\theta \cdot 4\cos\theta\,d\theta$$

$$= 16\int_0^{\pi/6} \cos^2\theta\,d\theta$$

$\therefore$ $\cos 2\theta = 2\cos^2\theta - 1$

$2\cos^2\theta = 1 + \cos 2\theta$

$$= 8\int_0^{\pi/6} (1+\cos 2\theta)\,d\theta$$

$$= 8\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/6}$$

$$= 8\left(\left(\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3}\right) - 0\right)$$

$$= \underline{\underline{\frac{4\pi}{3} + 2\sqrt{3}}}$$

2011

Q1 – 5 marks

Express $\dfrac{13-x}{x^2+4x-5}$ in partial fractions and hence obtain

$$\int \frac{13-x}{x^2+4x-5}\,dx.$$

Written Solutions

$$\frac{13-x}{x^2+4x-5} = \frac{13-x}{(x-1)(x+5)} = \frac{A}{x-1} + \frac{B}{x+5}$$

$$\Rightarrow \quad A(x+5) + B(x-1) = 13-x$$

let $x = 1$ $\qquad 6A = 12$

$$A = 2$$

let $x = -5$ $\qquad -6B = 18$

$$B = -3$$

So, $\underline{\underline{\dfrac{13-x}{x^2+4x-5} = \dfrac{2}{x-1} - \dfrac{3}{x+5}}}$

$$\Rightarrow \int \frac{13-x}{x^2+4x-5} = \int \left(\frac{2}{x-1} - \frac{3}{x+5}\right) dx$$

$$= \underline{\underline{2\ln|x-1| - 3\ln|x+5| + c}}$$

2011

Q11 – 7 marks

(*a*) Obtain the exact value of $\int_0^{\pi/4}(\sec x - x)(\sec x + x)dx$. **3**

(*b*) Find $\int \frac{x}{\sqrt{1-49x^4}}dx$. **4**

Written Solutions

a) $\int_0^{\frac{\pi}{4}} (\sec x - x)(\sec x + x)\ dx$

$= \int_0^{\frac{\pi}{4}} (\sec^2 x - x^2)\, dx$

$= \left[\tan x - \frac{1}{3}x^3\right]_0^{\frac{\pi}{4}}$

$= \left(\tan\frac{\pi}{4} - \frac{1}{3}\left(\frac{\pi}{4}\right)^3\right) - 0$

$= \underline{\underline{1 - \frac{\pi^3}{192}}}$

b) $\int \frac{x}{\sqrt{1-49x^4}}\ dx$ let $u = 7x^2$

$\frac{du}{dx} = 14x \Rightarrow dx = \frac{du}{14x}$

$= \int \frac{x}{\sqrt{1-u^2}} \cdot \frac{du}{14x}$

$= \frac{1}{14}\int \frac{du}{\sqrt{1-u^2}}$

$= \frac{1}{14}\sin^{-1} u + c$

$= \underline{\underline{\frac{1}{14}\sin^{-1}(7x^2) + c}}$

2010

Q3a – 3 marks

(*a*) Use the substitution $t = x^4$ to obtain $\int \frac{x^3}{1+x^8}\,dx$. **3**

Written Solutions

$$\int \frac{x^3}{1+x^8}\,dx \qquad \text{let } t = x^4$$

$$\frac{dt}{dx} = 4x^3$$

$$dx = \frac{dt}{4x^3}$$

So, $= \int \frac{x^3}{1+t^2} \cdot \frac{dt}{4x^3}$

$$= \frac{1}{4}\int \frac{dt}{1+t^2}$$

$$= \frac{1}{4}\tan^{-1} t + c$$

$$= \frac{1}{4}\tan^{-1} x^4 + c$$

2010

Q7 – 6 marks

Evaluate

$$\int_1^2 \frac{3x+5}{(x+1)(x+2)(x+3)}\,dx$$

expressing your answer in the form $\ln\frac{a}{b}$, where a and b are integers.

Written Solutions

$$\int_1^2 \frac{3x+5}{(x+1)(x+2)(x+3)}\,dx$$

let $\dfrac{3x+5}{(x+1)(x+2)(x+3)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} + \dfrac{C}{x+3}$

$\Rightarrow \quad A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2) = 3x+5$

let $x = -1$ .... $\Rightarrow$ $A = 1$

let $x = -2$ .... $\Rightarrow$ $B = 1$

let $x = -3$ .... $\Rightarrow$ $C = -2$

So, $\displaystyle\int_1^2 \left(\frac{1}{x+1} + \frac{1}{x+2} - \frac{2}{x+3}\right) dx$

$= \Big[\ln(x+1) + \ln(x+2) - 2\ln(x+3)\Big]_1^2$

$= \left[\ln\dfrac{(x+1)(x+2)}{(x+3)^2}\right]_1^2$

$= \ln\dfrac{12}{25} - \ln\dfrac{6}{16}$

$= \ln\left(\dfrac{12}{25} \div \dfrac{6}{16}\right)$

$= \underline{\underline{\ln\dfrac{32}{25}}}$

2010

Q15 – 10 marks

A new board game has been invented and the symmetrical design on the board is made from four identical "petal" shapes. One of these petals is the region enclosed between the curves $y = x^2$ and $y^2 = 8x$ as shown shaded in diagram 1 below.

Calculate the area of the complete design, as shown in diagram 2. **5**

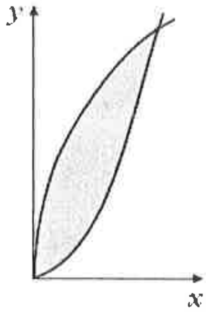


Diagram 1

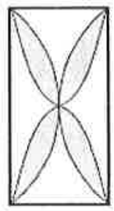
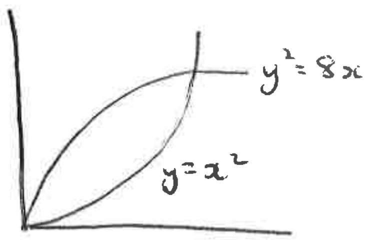


Diagram 2

The counter used in the game is formed by rotating the shaded area shown in diagram 1 above, through 360° about the y-axis. Find the volume of plastic required to make one counter. **5**

Written Solutions

$y = x^2$ ① $\quad y^2 = 8x$ ②



Subst ① in ②

$(x^2)^2 = 8x$

$x^4 - 8x = 0$

$x(x^3 - 8) = 0$

$\Rightarrow x = 0, \quad x = \sqrt[3]{8} = 2$

$$\text{Area} = 4\int_0^2 \left(2\sqrt{2}\,x^{\frac{1}{2}} - x^2\right) dx$$

$\therefore y = \sqrt{8x}$
$= 2\sqrt{2}\,x^{\frac{1}{2}}$

$$= 4\left[\frac{4}{3}\sqrt{2}\,x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^2$$

$$= 4\left[\left(\frac{4}{3}\sqrt{2}\cdot 2^{\frac{3}{2}} - \frac{1}{3}\cdot 2^3\right) - (0)\right]$$

$$= \frac{32}{3}\ \text{units}^2$$

PTO

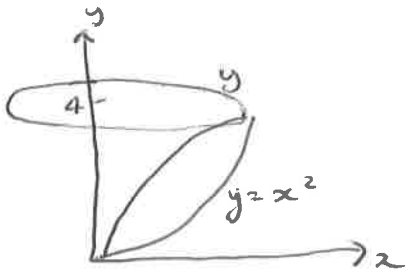
$$V = \pi \int x^2 \, dy$$

$$= \pi \int_0^4 \left(y - \frac{1}{64} y^4 \right) dy$$

$$= \pi \left[\frac{1}{2} y^2 - \frac{1}{320} y^5 \right]_0^4$$

$$= \frac{\pi}{2} \left(4^2 - \frac{1}{160} (4)^5 \right)$$

$$= \frac{24\pi}{5} \text{ units}^3$$



$\therefore$ (i) $y = 2^2$

$= 4$

(ii) $y^2 = 8x$

$\therefore x = \frac{1}{8} y^2$

2009

Q5 – 4 marks

Show that

$$\int_{\ln\frac{3}{2}}^{\ln 2} \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \ln\frac{9}{5}. \qquad 4$$

Written Solutions

$$\int_{\ln\frac{3}{2}}^{\ln 2} \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$$

let $u = e^x - e^{-x}$

$\frac{du}{dx} = e^x + e^{-x}$

$dx = \frac{du}{e^x + e^{-x}}$

LIMITS:

$x = \ln 2 \quad u = e^{\ln 2} - e^{-\ln 2}$

$= 2 - \frac{1}{2}$

$= \frac{3}{2}$

$x = \ln\frac{3}{2} \quad u = e^{\ln\frac{3}{2}} - e^{-\ln\frac{3}{2}}$

$= \frac{3}{2} - \frac{2}{3}$

$= \frac{5}{6}$

$$= \int_{\frac{5}{6}}^{\frac{3}{2}} \frac{e^x + e^{-x}}{u} \cdot \frac{du}{e^x + e^{-x}}$$

$$= \int_{\frac{5}{6}}^{\frac{3}{2}} \frac{du}{u}$$

$$= \Big[\ln u\Big]_{5/6}^{3/2}$$

$$= \ln\frac{3}{2} - \ln\frac{5}{6}$$

$$= \underline{\underline{\ln\frac{9}{5}}}$$

<u>OR</u> since $\frac{d}{dx}(e^x - e^{-x}) = e^x + e^{-x}$

$$\Rightarrow I = \Big[\ln|e^x - e^{-x}|\Big]_{\ln 3/2}^{\ln 2}$$

$$= \underline{\underline{\ln\frac{9}{5}}}$$

2009

Q7 – 6 marks

Use the substitution $x = 2\sin\theta$ to obtain the exact value of $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx$. **6**

(Note that $\cos 2A = 1 - 2\sin^2 A$.)

Written Solutions

$$\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx$$

let $x = 2\sin\theta$

$$\frac{dx}{d\theta} = 2\cos\theta$$

$$dx = 2\cos\theta \cdot d\theta$$

LIMITS

$x = 0 \Rightarrow 2\sin\theta = 0$

$\sin\theta = 0$

$x = \sqrt{2} \Rightarrow 2\sin\theta = \sqrt{2}$

$\theta = \frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} \frac{4\sin^2\theta \cdot 2\cos\theta\, d\theta}{2\cos\theta}$$

$$= \int_0^{\frac{\pi}{4}} 4\sin^2\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} (1-\cos 2\theta)\, d\theta$$

$$= 2\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{2} - 1$$

co ω

$$\sqrt{4-x^2}$$

$$= \sqrt{4-4\sin^2\theta}$$

$$= \sqrt{4(1-\sin^2\theta)}$$

$$= 2\cos\theta$$

2007

Q10 – 6 marks

Use the substitution $u = 1 + x^2$ to obtain $\int_0^1 \frac{x^3}{(1+x^2)^4}\,dx$. **5**

A solid is formed by rotating the curve $y = \frac{x^{3/2}}{(1+x^2)^2}$ between $x = 0$ and $x = 1$ through $360°$ about the x-axis. Write down the volume of this solid. **1**

Written Solutions

$$\int_0^1 \frac{x^3}{(1+x^2)^4}\,dx$$

let $u = 1 + x^2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

LIMITS

$x = 0 \Rightarrow u = 1$

$x = 1 \Rightarrow u = 2$

$$= \int_1^2 \frac{x^3}{u^4} \cdot \frac{du}{2x}$$

$\therefore 1 + x^2 = u$

$x^2 = u - 1$

$$= \frac{1}{2}\int_1^2 \frac{u-1}{u^4}\,du$$

$$= \frac{1}{2}\int_1^2 (u^{-3} - u^{-4})\,du$$

$$= \frac{1}{2}\left[\frac{u^{-2}}{-2} + \frac{u^{-3}}{3}\right]_1^2$$

$$= \frac{1}{24}$$