Maclaurin Theorem

2012

Q6 – 5 marks

Write down the Maclaurin expansion of e^x as far as the term in x^3.	1
Hence, or otherwise, obtain the Maclaurin expansion of $(1 + e^x)^2$ as far as the term in x^3.	4

Marking Instructions

Method 1

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$	1
$(1 + e^x)^2 = 1 + 2e^x + e^{2x}$	1M
$= 1 + 2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right) + \left(1 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{6} + \ldots\right)$	1
$= 1 + 2 + 2x + x^2 + \frac{1}{3}x^3 + 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \ldots$	1
$= 4 + 4x + 3x^2 + \frac{5}{3}x^3 + \ldots$	1

Method 2

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$	1
$(1 + e^x) = 2 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$	1
$(1 + e^x)^2 = \left(2 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(2 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)$	1M
$= 4 + 4x + 3x^2 + \frac{1}{3}x^3 + \frac{1}{2}x^3 + \frac{1}{2}x^3 + \frac{1}{3}x^3 + \ldots$	1
$= 4 + 4x + 3x^2 + \frac{5}{3}x^3 + \ldots$	1

Method 3

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$		1	
$f(x) = (1 + e^x)^2$	$f(0) = 4$		
$f'(x) = 2e^x(1 + e^x) = 2e^x + 2e^{2x}$	$f'(0) = 4$	1	can award marks for correct columns.
$f''(x) = 2e^x + 4e^{2x}$	$f''(0) = 6$	1	
$f'''(x) = 2e^x + 8e^{2x}$	$f'''(0) = 10$	1	
$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{6} + \ldots$			
$(1 + e^x)^2 = 4 + 4x + 3x^2 + \frac{5}{3}x^3 + \ldots$		1	

2011

Q5 – 6 marks

Obtain the first four terms in the Maclaurin series of $\sqrt{1+x}$, and hence write down the first four terms in the Maclaurin series of $\sqrt{1+x^2}$. **4**

Hence obtain the first four terms in the Maclaurin series of $\sqrt{(1+x)(1+x^2)}$. **2**

Marking Instructions

Let $f(x) = (1+x)^{\frac{1}{2}}$, then

$f(x) = (1+x)^{\frac{1}{2}} \Rightarrow f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} \Rightarrow f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}} \Rightarrow f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}} \Rightarrow f'''(0) = \frac{3}{8}$

1 for derivatives
1 for values

Hence

$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{4} \times \frac{x^2}{2} + \frac{3}{8} \times \frac{x^3}{6} - \ldots$ 1

$= 1 + \frac{1}{2}x - \frac{x^2}{8} + \frac{x^3}{16} - \ldots$

and replacing x by x^2 gives

$\sqrt{1+x^2} = 1 + \frac{1}{2}x^2 - \frac{x^4}{8} + \frac{x^6}{16} - \ldots$ 1

Thus

$\sqrt{(1+x)(1+x^2)} =$

$\left(1 + \frac{1}{2}x - \frac{x^2}{8} + \frac{x^3}{16} - \ldots\right)\left(1 + \frac{1}{2}x^2 - \frac{x^4}{8} + \frac{x^6}{16} - \ldots\right)$ **1M** for multiplying

$= 1 + \frac{1}{2}x + \frac{1}{2}x^2 - \frac{1}{8}x^2 + \frac{1}{4}x^3 + \frac{1}{16}x^3 + \ldots$

$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots$ 1

2010

Q9 – 4 marks

Obtain the first three non-zero terms in the Maclaurin expansion of $(1 + \sin^2 x)$. **4**

Marking Instructions

Let $f(x) = (1 + \sin^2 x)$. Then		
$f(0) = 1$	1	
$f'(x) = 2\sin x \cos x \Rightarrow f'(0) = 0$		
$= \sin 2x$		
$f''(x) = 2\cos 2x \Rightarrow f''(0) = 2$	1	
$f'''(x) = -4\sin 2x \Rightarrow f'''(0) = 0$		
$f''''(x) = -8\cos 2x \Rightarrow f''''(0) = -8$	1	one for each non-zero term
$f(x) = 1 + 2\frac{x^2}{2!} - 8\frac{x^4}{4!} + \ldots$	1	
$= 1 + x^2 - \frac{1}{3}x^4 + \ldots$		
Alternative 1		
$f(0) = 1$	1	
$f'(x) = 2\sin x \cos x \Rightarrow f'(0) = 0$		
$f''(x) = 2\cos^2 x - 2\sin^2 x \Rightarrow f''(0) = 2$	1	
$f'''(x) = 4(-\sin x)\cos x$ $-4\cos x \sin x \Rightarrow f'''(0) = 0$		
$f''''(x) = -8\cos^2 x + 8\sin^2 x \Rightarrow f''''(0) = -8$	1	
etc		
Alternative 2		
$f(x) = (1 + \sin^2 x)$		
$= 1 + \frac{1}{2} - \frac{1}{2}\cos 2x$	1	introducing cos2x
$= \frac{1}{2}(3 - \cos 2x)$		
$= \frac{1}{2}\left(3 - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \ldots\right)\right)$	1	expanding cos2x
$= \frac{1}{2}(3 - 1 + 2x - \frac{2}{3}x^4 - \ldots)$	1	simplifying
$= 1 + x^2 - \frac{1}{3}x^4 - \ldots$	1	finishing

2009

Q14 – 9 marks

Express $\frac{x^2+6x-4}{(x+2)^2(x-4)}$ in partial fractions. **4**

Hence, or otherwise, obtain the first three non-zero terms in the Maclaurin expansion of $\frac{x^2+6x-4}{(x+2)^2(x-4)}$. **5**

Marking Instructions

$$\frac{x^2+6x-4}{(x+2)^2(x-4)} = \frac{A}{(x+2)^2} + \frac{B}{x+2} + \frac{C}{x-4}$$ **M1**

$$x^2+6x-4 = A(x-4) + B(x+2)(x-4) + C(x+2)^2$$

Let $x = -2$ then $4 - 12 - 4 = -6A \Rightarrow A = 2$. **1**

Let $x = 4$ then $16 + 24 - 4 = 36C \Rightarrow C = 1$. **1**

Let $x = 0$ then

$-4 = -4A - 8B + 4C \Rightarrow -4 = -8 - 8B + 4 \Rightarrow B = 0$. **1**

Thus

$$\frac{x^2+6x-4}{(x+2)^2(x-4)} = \frac{2}{(x+2)^2} + \frac{1}{x-4}.$$

Let $f(x) = 2(x+2)^{-2} + (x-4)^{-1}$ then

$f(x) = 2(x+2)^{-2} + (x-4)^{-1} \Rightarrow f(0) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ **1**

$f'(x) = -4(x+2)^{-3} - (x-4)^{-2} \Rightarrow f'(0) = -\frac{1}{2} - \frac{1}{16} = -\frac{9}{16}$ **1**

$f''(x) = 12(x+2)^{-4} + 2(x-4)^{-3} \Rightarrow f''(0) = \frac{3}{4} - \frac{1}{32} = \frac{23}{32}$ **1**

Thus

$$\frac{x^2+6x-4}{(x+2)^2(x-4)} = \frac{1}{4} - \frac{9x}{16} + \frac{23x^2}{64} + \ldots$$ **2E1**

2008

Q12 – 7 marks

Obtain the first three non-zero terms in the Maclaurin expansion of $x\ln(2+x)$. **3**

Hence, or otherwise, deduce the first three non-zero terms in the Maclaurin expansion of $x\ln(2-x)$. **2**

Hence obtain the first **two** non-zero terms in the Maclaurin expansion of $x\ln(4-x^2)$. **2**

[*Throughout this question, it can be assumed that* $-2 < x < 2$.]

Marking Instructions

$f(x) = x\ln(2+x)$ so

$$f'(x) = \frac{x}{2+x} + \ln(2+x),\ f''(x) = \frac{2}{(2+x)^2} + \frac{1}{2+x},$$

$$f'''(x) = -\frac{4}{(2+x)^3} - \frac{1}{(2+x)^2}$$ 1

and so $f'(0) = \ln 2,\ f''(0) = 1,\ f'''(0) = -\frac{3}{4}$. 1

Thus $f(x) = (\ln 2)x + \frac{x^2}{2} - \frac{x^3}{8} + \dots$ 1

$x\ln(2-x) = -f(-x)$ 1

$= (\ln 2)x - \frac{x^2}{2} - \frac{x^3}{8} + \dots$ 1

$x\ln(4-x^2) = x\ln(2+x) + x\ln(2-x)$ 1

$= (2\ln 2)x - \frac{x^3}{4} + \dots$ 1

Alternative for first three marks:

$f(x) = x\ln(2+x) = x\left(\ln 2 + \ln\left(1+\frac{x}{2}\right)\right)$ 1

$= x\left(\ln 2 + \frac{x}{2} - \frac{x^2}{8} + \dots\right)$ 1

$= x\ln 2 + \frac{x^2}{2} - \frac{x^3}{8} + \dots$ 1

2007

Q6 – 5 marks

Find the Maclaurin series for $\cos x$ as far as the term in x^4.	2
Deduce the Maclaurin series for $f(x) = \frac{1}{2}\cos 2x$ as far as the term in x^4.	2
Hence write down the first three non-zero terms of the series for $f(3x)$.	1

Marking Instructions

$f(x) = f'(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$	1
$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots$	1
$f(x) = \frac{1}{2}\left(1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24} - \ldots\right)$	1
$\quad = \frac{1}{2} - x^2 + \frac{x^4}{3} - \ldots$	1
$f(3x) = \frac{1}{2} - (3x)^2 + \frac{1}{3}(3x)^4$	
$\quad = \frac{1}{2} - 9x^2 + 27x^4 - \ldots$	1

Alternative for third and fourth marks:

$f(x) = \frac{1}{2}\cos 2x$	$f(0) = \frac{1}{2}$
$f'(x) = -\sin 2x$	$f'(0) = 0$
$f''(x) = -2\cos 2x$	$f''(0) = -2$
$f'''(x) = 4\sin 2x$	$f'''(0) = 0$
$f''''(x) = 8\cos 2x$	$f''''(0) = 8$

1

In general

$$f(x) = f(0) + f'(0).x + f''(0)\frac{x^2}{2} + \ldots$$

Hence

$$f(x) = \frac{1}{2} + 0 + (-2)\frac{x^2}{2} + 0 + 8\frac{x^4}{24} + \ldots$$

$$= \frac{1}{2} - x^2 + \frac{x^4}{3} - \ldots \qquad 1$$