Further Sequences and Series

Power Series

Any series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + ...$$

is called a Power Series.

In many cases, the sum of such a series becomes bigger and bigger as you add on successive terms - i.e. the series diverges.

It is the sum of some series that converges to a particular limit that we will meet at Advanced Higher.

Maclaurin Series

Under certain circumstances a function *f(x)* is given by:

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + ... + \frac{f^{(n)}(0)x^n}{n!}$$

Note/ (1) Maclaurins theorem holds for functions for which $f^n(0)$ exists $\forall$ n

(2) The resulting power series must be convergent. The interval of congruence is known as the Domain of Validity.

Example

Use Maclaurins theorem to expand e^x as a series of ascending powers of x.

$f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad f''(0) = 1$

Hence $\quad f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)x^n}{n!}$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

Q: How far do you go?!

A: The question will always tell you e.g. "Write down as far as the x^3 term" or "Obtain the first four (non-zero) terms"

Example

Write down the first 3 non-zero terms of *sinx*

$$f(x) = \sin x \qquad f(0) = 0$$
$$f'(x) = \cos x \qquad f'(0) = 1$$
$$f''(x) = -\sin x \qquad f''(0) = 0$$
$$f'''(x) = -\cos x \qquad f'''(0) = -1$$
$$f^{IV}(x) = \sin x \qquad f^{IV}(0) = 0$$
$$f^{V}(x) = \cos x \qquad f^{V}(0) = 1$$

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$$f^{V}(x) = \cos x \qquad f^{V}(0) = 1$$

Hence $f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)x^n}{n!}$

$$\sin x = 0 + \frac{(1)x^1}{1!} + \frac{(0)x^2}{2!} + \frac{(-1)x^3}{3!} + \frac{(0)x^4}{4!} + \frac{(1)x^5}{5!}$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

Example

Write down the first 3 non-zero terms of $\ln(1+x)$

$$f(x) = \ln(1+x) \qquad f(0) = 0$$
$$f'(x) = \frac{1}{1+x} \qquad f'(0) = 1$$
$$f''(x) = -\frac{1}{(1+x)^2} \qquad f''(0) = -1$$
$$f'''(x) = \frac{2}{(1+x)^3} \qquad f'''(0) = 2$$

Hence $f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)x^n}{n!}$

$$\ln(1+x) = 0 + \frac{(1)x^1}{1!} + \frac{(-1)x^2}{2!} + \frac{(2)x^3}{3!}$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

The following list gives some standard results which are worth remembering. You are expected to be able to derive them.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

DON'T FORGET

Try to use standard results as much as possible.

Top Tip

If the questions says "Write down" you can use these standard results directly. If the questions says "Obtain" you must derive them (unless they are part of an associated series - next).

LET'S THINK ABOUT THIS

Why does the Maclaurin series for cosx only ctain even powers, while that for sinx only contains odd powers?

A: cosx is an even function of x, and sinx is an odd function of x.

Associated Series - Using more than one expansion

Example

Obtain the Maclaurin expansion for $f(x) = e^{-2x}sin3x$ as far as x^4 using the expansion for e^x and $sinx$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$e^{-2x} = 1 + \frac{(-2x)}{1!} + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(-2x)^4}{4!}$$

$$e^{-2x} = 1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3}$$

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$sin3x = 3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} + \dots$$

$$sin3x = 3x - \frac{9x^3}{2} + \frac{81x^5}{40} + \dots$$

Hence

$$f(x) = e^{-2x}sin3x = \left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3}\right)\left(3x - \frac{9x^3}{2} + \frac{81x^5}{40} + \dots\right)$$

Q: Why only expand e^{-2x} to the power 4 and $sin3x$ to the power 5?

Hence

$$f(x) = e^{-2x} sin3x = \left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3}\right)\left(3x - \frac{9x^3}{2} + \frac{81x^5}{40} + \dots\right)$$

$$= 3x - \frac{9x^3}{2} - 6x^2 + \frac{18x^4}{2} + 6x^3 - \frac{12x^4}{3} + \dots$$

$$= 3x - 6x^2 + \frac{3x^3}{2} + 5x^4$$

We could have used the Product Rule to differentiate f(x) to obtain the Maclaurin expansion as far as x^4.

Example

Obtain the Maclaurin expansion for $f(x) = \ln(cosx)$ as far as the term in x^6

The only logarithmic expansion is that for $\ln(1 + x)$ $\therefore$ we need to write $cosx$ in the form (1 + ...)

The easiest method is to use $cosx = [1 + (cosx - 1)]$

So, $\ln(cosx) = ln[1 + (cosx - 1)]$

$$ln[1 + (cosx - 1)] = (cosx - 1) - \frac{(cosx-1)^2}{2} + \frac{(cosx-1)^3}{3} - \dots$$

But $cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$$cosx - 1 = -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

And $\ln(cosx) =$

$$\left[-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right] - \frac{1}{2}\left[-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right]^2 + \frac{1}{3}\left[-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right]^3$$

$$= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^4}{8} + \frac{x^6}{96} + \frac{x^6}{96} - \frac{x^6}{24}$$

$$= -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45}$$