2012

Q14 – 9 marks

(*a*) Use Gaussian elimination to obtain the solution of the following system of equations in terms of the parameter λ.

$$4x + 6z = 1$$
$$2x - 2y + 4z = -1$$
$$-x + y + \lambda z = 2 \qquad 5$$

(*b*) Describe what happens when $\lambda = -2$. 1

(*c*) When $\lambda = -1 \cdot 9$ the solution is $x = -22 \cdot 25$, $y = 8 \cdot 25$, $z = 15$.

Find the solution when $\lambda = -2 \cdot 1$. 2

Comment on these solutions. 1

Written Solutions

(a) $\left(\begin{array}{ccc|c} 4 & 0 & 6 & 1 \\ 2 & -2 & 4 & -1 \\ -1 & 1 & \lambda & 2 \end{array}\right)$

$\begin{array}{c} \\ 2R_2 - R_1 \\ 4R_3 + R_1 \end{array} \left(\begin{array}{ccc|c} 4 & 0 & 6 & 1 \\ 0 & -4 & 2 & -3 \\ 0 & 4 & 4\lambda+6 & 9 \end{array}\right)$

$(\times -1) \left(\begin{array}{ccc|c} 4 & 0 & 6 & 1 \\ 0 & 4 & -2 & 3 \end{array}\right)$

$\begin{array}{c} \\ \\ R_3 + R_2 \end{array} \left(\begin{array}{ccc|c} 4 & 0 & 6 & 1 \\ 0 & 4 & -2 & 3 \\ 0 & 0 & 4\lambda+8 & 6 \end{array}\right)$

① $R_3 = (4\lambda+8)z = 6$

$z = \dfrac{6}{4\lambda+8} = \dfrac{3}{2\lambda+4}$

② $R_2 = 4y - 2z = 3$

$4y - \dfrac{6}{2\lambda+4} = 3$

$4y = \dfrac{3(2\lambda+4)+6}{2\lambda+4}$

$4y = \dfrac{6\lambda+18}{2\lambda+4}$

$y = \dfrac{6\lambda+18}{4(2\lambda+4)}$

$= \dfrac{3\lambda+9}{4(\lambda+2)}$

③ $4x + 6z = 1$

$4x + \dfrac{18}{2\lambda+4} = 1$

$4x = \dfrac{2\lambda+4-18}{2\lambda+4}$

$4x = \dfrac{2\lambda-14}{2\lambda+4}$

$4x = \dfrac{\lambda-7}{\lambda+2}$

$x = \dfrac{\lambda-7}{4(\lambda+2)}$

So, $x = \dfrac{\lambda-7}{4(\lambda+2)}$, $y = \dfrac{3\lambda+9}{4(\lambda+2)}$, $z = \dfrac{3}{2\lambda+4}$

Matrices

2013

Q3 – 5 marks

Matrices A and B are defined by $A=\begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix}$.

(*a*) Find A^2. **1**

(*b*) Find the value of p for which A^2 is singular. **2**

(*c*) Find the values of p and x if $B = 3A'$. **2**

Written Solutions

(a) $A^2 = \begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 16-2p & 5p \\ -10 & -2p+1 \end{pmatrix}}}$

(b) $\det A^2 = ad - bc = 0$ for A^2 to be singular i.e. no inverse

$$(16-2p)(-2p+1) - 5p(-10) = 0$$
$$-32p + 16 + 4p^2 - 2p + 50p = 0$$
$$4p^2 + 16p + 16 = 0$$
$$p^2 + 4p + 4 = 0$$
$$(p+2)(p+2) = 0$$

So $\underline{\underline{p = -2}}$

(c) $A' = \begin{pmatrix} 4 & -2 \\ p & 1 \end{pmatrix}$

$$B = 3A'$$

$$\begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 12 & -6 \\ 3p & 3 \end{pmatrix}$$

$\Rightarrow$ $\underline{\underline{x = 12}}$ and $3p = 1$

$\underline{\underline{p = \frac{1}{3}}}$

2012

Q9 – 4 marks

A non-singular $n \times n$ matrix A satisfies the equation $A + A^{-1} = I$, where I is the $n \times n$ identity matrix. Show that $A^3 = kI$ and state the value of k.

Written Solutions

$$A + A^{-1} = I$$

$(\times A)$ $$A^2 + I = A$$ using $A + A^{-1} = I$, $A = I - A^{-1}$

$$A^2 + I = I - A^{-1}$$

$$A^2 = -A^{-1}$$

$(\times A)$ $$A^3 = -I$$

i.e. $\underline{\underline{k = -1}}$

(b) When $\lambda = -2$ the final row $(8 + 4\lambda) z = 6$ would give

$0z = 6$ which is inconsistent.

There are no sol^n.

(c) When $\lambda = -2.1$ (subst into $x =$, $y =$ and $z =$)

$x = 22.75$, $y = -6.75$, $z = -15$

Although the values of λ are close, the values of x, y and z are quite different. The system is ill-conditioned near $\lambda = -2$.

2011

Q4 – 6 marks

(*a*) For what value of λ is $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ -1 & \lambda & 6 \end{pmatrix}$ singular?

(*b*) For $A = \begin{pmatrix} 2 & 2\alpha-\beta & -1 \\ 3\alpha+2\beta & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}$, obtain values of α and β such that

$$A' = \begin{pmatrix} 2 & -5 & -1 \\ -1 & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}.$$

Written Solutions

(a) Singular when the determinant is 0.

$$\Rightarrow \quad 1\begin{vmatrix} 0 & 2 \\ \lambda & 6 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ -1 & 6 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ -1 & \lambda \end{vmatrix}$$

$$= 1(0-2\lambda) - 2(18-(-2)) - 1(3\lambda - 0)$$

$$= -2\lambda - 40 - 3\lambda$$

$$= -5\lambda - 40$$

So, for singularity, $-5\lambda - 40 = 0$

$$-5\lambda = 40$$

$$\underline{\underline{\lambda = -8}}$$

(b) $$A' = \begin{pmatrix} 2 & \boxed{3\alpha+2\beta} & -1 \\ \boxed{2\alpha-\beta} & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & \boxed{-5} & -1 \\ \boxed{-1} & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}$$

Hence, by comparing matrices:

$2\alpha - \beta = -1$ ①

$3\alpha + 2\beta = -5$ ②

... solving simultaneously gives:

$\underline{\underline{\alpha = -1}}$ and $\underline{\underline{\beta = -1}}$

2010

Q4 – 4 marks

Obtain the 2×2 matrix M associated with an enlargement, scale factor 2, followed by a clockwise rotation of $60°$ about the origin.

Written Solutions

Scale Factor 2:

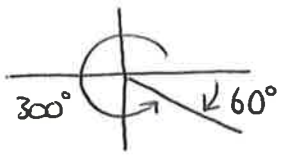
$(x, y) \rightarrow (x', y') = (2x, 2y)$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix} = \begin{pmatrix} 2x + 0y \\ 0x + 2y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

So, $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is the matrix associated with an enlargement.

Rotation:

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ represents anticlockwise rotation



So, $\begin{pmatrix} \cos 300 & -\sin 300 \\ \sin 300 & \cos 300 \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$M = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$

2010

Q14 – 10 marks

Use Gaussian elimination to show that the set of equations

$$x - y + z = 1$$
$$x + y + 2z = 0$$
$$2x - y + az = 2$$

has a unique solution when $a \neq 2{\cdot}5$. 5

Explain what happens when $a = 2{\cdot}5$. 1

Obtain the solution when $a = 3$. 1

Given $A = \begin{pmatrix} 5 & 2 & -3 \\ 1 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, calculate AB. 1

Hence, or otherwise, state the relationship between A and the matrix

$$C = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}.$$ 2

Written Solutions

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & -1 & a & 2 \end{array}\right)$$

$$\begin{array}{l} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 1 & a-2 & 0 \end{array}\right)$$

$$\begin{array}{l} \\ \\ 2R_3 - R_2 \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 2a-5 & 1 \end{array}\right)$$

① $R_3 : (2a-5)z = 1$

$z = \dfrac{1}{2a-5}$

② $R_2 : 2y + z = -1$

$2y + \dfrac{1}{2a-5} = -1$

$2y = \dfrac{-(2a-5) - 1}{2a-5}$

$= \dfrac{-2a+4}{2a-5}$

$y = \dfrac{-a+2}{2a-5}$

③ $R_1 : x - y + z = 1$

$x - \left(\dfrac{-a+2}{2a-5}\right) + \left(\dfrac{1}{2a-5}\right) = \dfrac{2a-5}{2a-5}$

$x - \left(\dfrac{-a+2}{2a-5}\right) = \dfrac{2a-6}{2a-5}$

$x = \dfrac{a-4}{2a-5}$

So, $x = \dfrac{a-4}{2a-5}$, $y = \dfrac{2-a}{2a-5}$, $z = \dfrac{1}{2a-5}$

and all exist when $2a-5 \neq 0$

i.e. $a \neq \frac{5}{2}$.

From the final row in the tableau, when $a = \frac{5}{2}$, there are no sol^n.

When $a = 3$, $x = -1$, $y = -1$, $z = 1$.

$$AB = \begin{pmatrix} 5 & 2 & -3 \\ 1 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}}}$$

From above, we have $C\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and also

$A\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ which suggests $AC = I$ and this

can be verified directly.

Hence $\underline{\underline{A \text{ is the inverse of } C}}$ (or vice versa).

2009

Q2 – 5 marks

Given the matrix $A = \begin{pmatrix} t+4 & 3t \\ 3 & 5 \end{pmatrix}$.

(*a*)	Find A^{-1} in terms of t when A is non-singular.	3
(*b*)	Write down the value of t such that A is singular.	1
(*c*)	Given that the transpose of A is $\begin{pmatrix} 6 & 3 \\ 6 & 5 \end{pmatrix}$, find t.	1

Written Solutions

(a) $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$= \dfrac{1}{20-4t}\begin{pmatrix} 5 & -3t \\ -3 & t+4 \end{pmatrix}$

$ad - bc$
$= (t+4)\times 5 - (3t)\times 3$
$= 5t + 20 - 9t$
$= 20 - 4t$

(b) Singular when $ad - bc = 0$.

So, $20 - 4t = 0$
$-4t = -20$
$t = 5$

(c) $A^{t} = \begin{pmatrix} \boxed{t+4} & 3 \\ 3t & 5 \end{pmatrix} = \begin{pmatrix} \boxed{6} & 3 \\ 6 & 5 \end{pmatrix}$

$\Rightarrow t + 4 = 6$
$t = 2$

2009

Q16(a) – 5 marks

Use Gaussian elimination to solve the following system of equations

$$\begin{aligned} x + y - z &= 6 \\ 2x - 3y + 2z &= 2 \\ -5x + 2y - 4z &= 1. \end{aligned}$$

Written Solutions

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2 & -3 & 2 & 2 \\ -5 & 2 & -4 & 1 \end{array}\right)$$

$$\begin{array}{c} \\ R_2-2R_1 \\ R_3+5R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & -5 & 4 & -10 \\ 0 & 7 & -9 & 31 \end{array}\right)$$

$$\begin{array}{c} \\ 7R_2 \\ 5R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & -35 & 28 & -70 \\ 0 & 35 & -45 & 155 \end{array}\right)$$

$$\begin{array}{c} \\ \\ R_2+R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & -35 & 28 & -70 \\ 0 & 0 & -17 & 85 \end{array}\right)$$

$$\begin{array}{c} \\ \frac{1}{7}R_2 \\ \frac{1}{17}R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & -5 & 4 & -10 \\ 0 & 0 & -1 & 5 \end{array}\right)$$

$$\therefore \quad -z = 5$$
$$z = -5$$

Since $z = -5$

$$\begin{aligned} -5y + 4z &= -10 \\ -5y - 20 &= -10 \\ y &= -2 \end{aligned}$$

Since $y = -2$, $z = -5$

$$\begin{aligned} x + y - z &= 6 \\ x - 2 + 5 &= 6 \\ x &= 3 \end{aligned}$$

Sol^n, is $\underline{\underline{x = 3}}$, $\underline{\underline{y = -2}}$, $\underline{\underline{z = -5}}$

2008

Q6 – 5 marks

Let the matrix $A = \begin{pmatrix} 1 & x \\ x & 4 \end{pmatrix}$.

(*a*) Obtain the value(s) of x for which A is singular.

(*b*) When $x = 2$, show that $A^2 = pA$ for some constant p.
Determine the value of q such that $A^4 = qA$.

Written Solutions

(a) Singular when $ad - bc = 0$

So, $1(4) - x(x) = 0$

$4 - x^2 = 0$

$x^2 = 4$

$\underline{\underline{x = \pm 2}}$

(b) $x = 2$: $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

$A^2 = A \times A$

$= \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

$= \begin{pmatrix} 5 & 10 \\ 10 & 20 \end{pmatrix}$

$= 5\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

$\therefore$ $\underline{\underline{A^2 = 5A}}$ i.e. $p = 5$

Thus, $A^4 = (A^2)^2 = (5A)^2 = 25A^2$

$= 25(5A)$

$= 125A$

$\therefore$ $\underline{\underline{q = 125}}$

2007

Q5 – 5 marks

Matrices A and B are defined by

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} x+2 & x-2 & x+3 \\ -4 & 4 & 2 \\ 2 & -2 & 3 \end{pmatrix}.$$

(*a*) Find the product AB.

(*b*) Obtain the determinants of A and of AB.

Hence, or otherwise, obtain an expression for det B.

Written Solutions

(a)
$$AB = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x+2 & x-2 & x+3 \\ -4 & 4 & 2 \\ 2 & -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} (x+2)-2 & (x-2)+2 & (x+3)-3 \\ -4-2 & 4+2 & 2-3 \\ -4+4 & 4-4 & 2+6 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} x & x & x \\ -6 & 6 & -1 \\ 0 & 0 & 8 \end{pmatrix}}}$$

(b)
$$\det AB = x\begin{vmatrix} 6 & -1 \\ 0 & 8 \end{vmatrix} - x\begin{vmatrix} -6 & -1 \\ 0 & 8 \end{vmatrix} + x\begin{vmatrix} -6 & 6 \\ 0 & 0 \end{vmatrix}$$

$$= x(48-0) - x(-48-0) + x(0-0)$$

$$= 48x + 48x + 0$$

$$= \underline{\underline{96x}}$$

$$\det A = 1\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - 0 - 1\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 1(2+1) - 1(0-0)$$

$$= \underline{\underline{3}}$$

$$\det AB = \det A \times \det B$$

$$\det B = \frac{\det AB}{\det A}$$

$$\det B = \underline{\underline{32x}}$$

2005

Q6 – 6 marks

Use Gaussian elimination to solve the system of equations below when $\lambda \neq 2$:

$$\begin{aligned} x &+ y + 2z = 1 \\ 2x &+ \lambda y + z = 0 \\ 3x &+ 3y + 9z = 5. \end{aligned}$$

Explain what happens when $\lambda = 2$.

Written Solutions

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & \lambda & 1 & 0 \\ 3 & 3 & 9 & 5 \end{array}\right) \xrightarrow{\substack{R_2-2R_1 \\ R_3-3R_1}} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & (\lambda-2) & -3 & -2 \\ 0 & 0 & 3 & 2 \end{array}\right)$$

So, $3z = 2$

$z = \frac{2}{3}$

since $z = \frac{2}{3}$

$$\begin{aligned} (\lambda-2)y - 3z &= -2 \\ (\lambda-2)y - 2 &= -2 \\ (\lambda-2)y &= 0 \\ y &= 0 \end{aligned}$$

since $z = \frac{2}{3}$, $y = 0$

$$\begin{aligned} x + y + 2z &= 1 \\ x + 0 + 2\left(\tfrac{2}{3}\right) &= 1 \\ x + \tfrac{4}{3} &= 1 \\ x &= -\tfrac{1}{3} \end{aligned}$$

Sol^n is $x = -\frac{1}{3}$, $y = 0$, $z = \frac{2}{3}$ when $\lambda \neq 2$

When $\lambda = 2$ (from 2nd matrix)
$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 3 & 2 \end{array}\right)$$

So, when $\lambda = 2$, the second and third rows are the same i.e. redundant.
Thus there is an infinite number of sol^n.

2005

Q7 – 6 marks

Given the matrix $A = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$, show that $A^2 + A = kI$ for some constant k, where I is the 3×3 unit matrix.

Obtain the values of p and q for which $A^{-1} = pA + qI$.

Written Solutions

$$A^2 + A = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}\begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} + \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -4 & -2 \\ -1 & 2 & -1 \\ 1 & 2 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 2I \quad \text{i.e.} \quad \underline{\underline{k = 2}}$$

$$A^{-1}(A^2 + A) = A^{-1}2I = 2A^{-1}$$

i.e.
$$2A^{-1} = A^{-1}(A^2 + A)$$
$$2A^{-1} = A^{-1}A^2 + A^{-1}A$$
$$2A^{-1} = A^1 + I$$
$$A^{-1} = \tfrac{1}{2}A + \tfrac{1}{2}I$$

i.e. $\underline{p = \tfrac{1}{2}}$ and $\underline{\underline{q = \tfrac{1}{2}}}$

2004

Q6 – 5 marks

Write down the 2×2 matrix M_1 associated with an anti-clockwise rotation of $\frac{\pi}{2}$ radians about the origin.

Write down the matrix M_2 associated with reflection in the x-axis.

Evaluate $M_2 M_1$ and describe geometrically the effect of the transformation represented by $M_2 M_1$.

Written Solutions

$M_1 \Rightarrow$ anti-clockwise rotation, $\frac{\pi}{2}$ radians

$M_2 \Rightarrow$ reflection in x-axis

$$M_1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

x-coordinate stays same, y changes.

$$M_2 M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \underline{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}$$

2003

A6 – 6 marks

Use elementary row operations to reduce the following system of equations to upper triangular form

$$\begin{aligned} x &+ y &+ 3z &= 1 \\ 3x &+ ay &+ z &= 1 \\ x &+ y &+ z &= -1. \end{aligned}$$

Hence express x, y and z in terms of the parameter a.

Explain what happens when $a = 3$.

Written Solutions

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 3 & a & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right)$$

$$\begin{array}{c} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & (a-3) & -8 & -2 \\ 0 & 0 & -2 & -2 \end{array}\right)$$

So, $-2z = -2$

$z = 1$

Since $z = 1$, $(a-3)y - 8z = -2$

$(a-3)y - 8 = -2$

$(a-3)y = 6$

$y = \dfrac{6}{a-3}$

Since $z = 1$, $y = \dfrac{6}{a-3}$

$x + y + 3z = 1$

$x + \dfrac{6}{a-3} + 3 = 1$

$x + \dfrac{6}{a-3} = -2$

$x = -2 - \dfrac{6}{a-3}$

Sol^n/ are $\underline{\underline{x = -2 - \dfrac{6}{a-3}}}$, $\underline{\underline{y = \dfrac{6}{a-3}}}$, $\underline{\underline{z = 1}}$

When $a = 3$ (from 2^{nd} matrix) $\left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 0 & -8 & -2 \\ 0 & 0 & -2 & -2 \end{array}\right)$

Rows 2 and 3 provide different sol^n/ i.e. $\underline{\underline{\text{inconsistent}}}$

2002

A1 – 5 marks

Use Gaussian elimination to solve the following system of equations

$$\begin{aligned} x + y + 3z &= 2 \\ 2x + y + z &= 2 \\ 3x + 2y + 5z &= 5. \end{aligned}$$

Written Solutions

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 2 & 1 & 1 & 2 \\ 3 & 2 & 5 & 5 \end{array}\right)$$

$$\begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & -1 & -5 & -2 \\ 0 & -1 & -4 & -1 \end{array}\right)$$

$$\begin{array}{l} \\ \\ R_3 - R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & -1 & -5 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

So, $z = 1$

Since $z = 1$

$$\begin{aligned} -y - 5z &= -2 \\ -y - 5 &= -2 \\ -y &= 3 \\ y &= -3 \end{aligned}$$

Since $y = -3$, $z = 1$

$$\begin{aligned} x + y + 3z &= 2 \\ x - 3 + 3 &= 2 \\ x &= 2 \end{aligned}$$

Sol^n is $\underline{\underline{x = 2}}$, $\underline{\underline{y = -3}}$, $\underline{\underline{z = 1}}$

2002

B4 – 4 marks

Write down the 2×2 matrix A representing a reflection in the x-axis and the 2×2 matrix B representing an anti-clockwise rotation of $30°$ about the origin.

Hence show that the image of a point (x, y) under the transformation A followed by the transformation B is $\left(\frac{kx+y}{2}, \frac{x-ky}{2}\right)$, stating the value of k.

Written Solutions

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$B \cdot A \cdot \underline{x} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{2}x + \frac{1}{2}y \\ \frac{1}{2}x - \frac{\sqrt{3}}{2}y \end{pmatrix}$$

Thus $\left(\frac{kx+y}{2}, \frac{x-ky}{2}\right) \Rightarrow \left(\frac{\sqrt{3}x+y}{2}, \frac{x-\sqrt{3}y}{2}\right)$ i.e. $\underline{\underline{k = \sqrt{3}}}$