SECOND ORDER

Differential Equations

Non-Homogeneous Second Order Differential Equations

To solve $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$

first solve the homogeneous equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

This gives the complementary function (**CF**).

We then find a particular integral (**PI**) *(methods given below)* which leads to the general solution of the non-homogeneous equation.

General solution = complementary function + particular integral

Finding Particular Integrals

1. If $f(x)$ is a polynomial function of degree n we try y = a polynomial function of degree n with constant coefficients.

   If $f(x) = x^2 + 2x - 1$ try $y = Cx^2 + Dx + E$

   If $f(x) = 2x+1$ try $y = Cx + D$

   If $f(x)=x^2-1$ try $y=Cx^2+Dx+E$ note that the term Dx must be included in the particular integral as $f(x)=x^2+0x-1$

   We then find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and then substitute into the equation.

   By equating coefficients of the left hand side and the right hand side we find the constants C, D etc

2. If $f(x)$ is an exponential function try y = an exponential function of equal power.

   If $f(x) = 3e^{2x}$ try $y = Ce^{2x}$

   Continue as in 1.

3. If $f(x)$ is a trigonometric function, then try y = a trig function.

   If $f(x) = 4\cos 2x$ try $y = C\sin 2x + D\cos 2x$. Note you need both cos and sin in your P.I. as $f(x)=0\sin 2x+4\cos 2x$

   If $f(x)=2\sin x+\cos x$ try $y=C\sin x+D\cos x$

   Again continue as in 1.

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

So when choosing the form of the **PI** we usualy choose the same form as the $f(x)$.

This reasoning leads us to select the **PI** according to the steps:

- try the same form as $f(x)$
- if this is the same form as a term of the **CF** then try the same form as $xf(x)$
- if this is the same form as a term of the **CF** then we try the same form as $x^2f(x)$.

Ex 1. Solve $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=4x+4$

A.E. $\Rightarrow \quad m^2+3m+2=0$ $\qquad (b^2-4ac=1)$

$(m+2)(m+1)=0$ $\qquad \Rightarrow m=-2,\ m=-1$

C.F. $y=Ae^{-2x}+Be^{-x}$

For the P.I. try $\quad y=Cx+D$

$$\frac{dy}{dx}=C$$

$$\frac{d^2y}{dx^2}=0$$

Substitute into

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=4x+4$$

$$0+3C+2(Cx+D)=4x+4$$

$$2Cx+3C+2D=4x+4$$

Equate coefficients of x $\quad \Rightarrow 2C=4$

$C=2$

Equate constants $\quad \Rightarrow 3C+2D=4$

$3(2)+2D=4$

$D=-1$

P.I. $y=2x-1$

So, G.S. = C.F. + P.I.

$$y=Ae^{-2x}+Be^{-x}+2x-1$$

Ex 2. Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 3e^{-2x}$

A.E $\Rightarrow m^2 + 2m + 2 = 0$ $\quad (b^2 - 4ac = -4)$ $\Rightarrow$ doesn't factorise

$$m = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$$

C.F. $y = e^{-x}(A\sin x + B\cos x)$

For a P.I.

$$y = Ce^{-2x}$$

$$\frac{dy}{dx} = -2Ce^{-2x}$$

$$\frac{d^2y}{dx^2} = 4Ce^{-2x}$$

Substitute into

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 3e^{-2x}$$

$$4Ce^{-2x} + 2(-2Ce^{-2x}) + 2Ce^{-2x} = 3e^{-2x}$$

$$2Ce^{-2x} = 3e^{-2x} \quad \Rightarrow C = \frac{3}{2}$$

P.I. $y = \frac{3}{2}e^{-2x}$

So, G.S. = C.F. + P.I.

$$y = e^{-x}(A\sin x + B\cos x) + \frac{3}{2}e^{-2x}$$

Ex 3. Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 30\sin 2x$

A.E. $\Rightarrow m^2 + 6m + 10 = 0$ $\quad (b^2 - 4ac = -4)$ $\Rightarrow$ doesn't factorise.

$$m = \frac{-6 \pm \sqrt{36-40}}{2} = \frac{-6 \pm \sqrt{-4}}{2} = -3 \pm i$$

C.F. $y = e^{-3x}(A\sin x + B\cos x)$

For the P.I try $y = \text{C} \sin 2x + \text{D} \cos 2x$

$$\frac{dy}{dx} = 2C\cos 2x - 2D\sin 2x$$

$$\frac{d^2y}{dx^2} = -4C\sin 2x - 4D\cos 2x$$

Substitute into

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 30\sin 2x$$

Substitute into

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 30\sin 2x$$

$$-4\text{C}\sin 2x - 4\text{D}\cos 2x + 6(2\text{C}\cos 2x - 2\text{D}\sin 2x) + 10(\text{C}\sin 2x + \text{D}\cos 2x) = 30 \sin 2x$$

$-4\text{D} + 12\text{C} + 10\text{D} = 0$ (equating cos 2x)
$-4\text{C} - 12\text{D} + 10\text{C} = 30$ (equating sin 2x)

$12\text{C} + 6\text{D} = 0$
$6\text{C} - 12\text{D} = 30$

solving simultaneously gives C = 1 and D = -2

P.I. = $\sin 2x - 2\cos 2x$

G.S. = C.F. + P.I.

$y = e^{-3x}(A\cos x + B\sin x) + \sin 2x - 2\cos 2x$