Proofs

2013

Q9 – 6 marks

Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n}\left(4r^3+3r^2+r\right)=n(n+1)^3$$ AIM. $(k+1)((k+1)+1)^3$ 6

Written Solutions

PI by induction – summation type

$n=1$: LHS: $4(1)^3+3(1)^2+1 = 8$ RHS: $1(1+1)^3 = 8$

LHS = RHS $\Rightarrow$ true for $n=1$

Assume true for $n=k$ $\Rightarrow$ $\sum_{r=1}^{k}(4r^3+3r^2+r) = k(k+1)^3$ – ① must be called into play during PI
(inductive hypothesis)

Consider $n=k+1$ $\Rightarrow$

$$\sum_{r=1}^{k+1}(4r^3+3r^2+r) = \sum_{r=1}^{k}(4r^3+3r^2+r) + \left(4(k+1)^3+3(k+1)^2+(k+1)\right)$$

$$= k(k+1)^3 + 4(k+1)^3 + 3(k+1)^2 + (k+1) \quad \text{using ①}$$

$$= (k+1)\left[k(k+1)^2 + 4(k+1)^2 + 3(k+1) + 1\right]$$

$$= (k+1)\left[(k+4)(k+1)^2 + 3k+4\right]$$

$$= (k+1)\left[k^3+6k^2+12k+8\right]$$

$$= (k+1)(k+2)^3 \quad \text{since } (k+2)^3 = k^3+3\cdot k^2\cdot 2+3k\cdot 2^2+2^3$$

$$= (k+1)((k+1)+1)^3$$ as ~~req~~ req'd for $n=k+1$

$\Rightarrow$ true for $n=k$ $\Rightarrow$ true for $n=k+1$
since true for $n=1$ $\Rightarrow$ by induction
true $\forall n \in \mathbb{Z}^+$

2013

Q12 – 4 marks

Let n be a natural number.
For each of the following statements, decide whether it is true or false.
If true, give a proof; if false, give a counterexample.

A If n is a multiple of 9 then so is n^2.

B If n^2 is a multiple of 9 then so is n. **4**

Written Solutions

A Let $n = 9k$ where $k \in \mathbb{N}$

$\Rightarrow n^2 = (9k)^2$

$= 9(9k^2)$

$\Rightarrow$ multiple of 9 since $9k^2 \in \mathbb{N}$.

$\Rightarrow$ statement A is true

B Let $n^2 = 9$ then $n = 3$

3 is not a multiple of 9

$\Rightarrow$ statement B is false

2012

Q16a – 6 marks

(*a*) Prove by induction that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

AIM $(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$

for all integers $n \geq 1$. 6

Written Solutions

$n=1$: LHS $= \cos\theta + i\sin\theta$ RHS $= \cos 1\theta + i\sin 1\theta$

$= \cos\theta + i\sin\theta$

$\Rightarrow$ LHS = RHS

$\Rightarrow$ true for $n=1$

Assume true for $n=k \Rightarrow (\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$ ①

Consider $n=k+1$: $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta).(\cos\theta + i\sin\theta)^k$

$= (\cos\theta + i\sin\theta)(\cos k\theta + i\sin k\theta)$ ①

$= \cos\theta\cos k\theta + \cos\theta\, i\sin k\theta$

$+ i\sin\theta\cos k\theta - \sin\theta\sin k\theta$

pull together real & imaginary

$= (\cos\theta\cos k\theta - \sin\theta\sin k\theta)$

$+ i(\sin\theta\cos k\theta + \cos\theta\sin k\theta)$

$= \cos(\theta + k\theta) + i(\sin\theta + k\theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

$\Rightarrow$ true for $n=k \Rightarrow$ true for $n=k+1$

$\Rightarrow$ since true for $n=1 \Rightarrow$ true $\forall n, n \in \mathbb{N}$.

2011

Q12 – 5 marks

Prove by induction that $8^n + 3^{n-2}$ is divisible by 5 for all integers $n \geq 2$. **5**

Written Solutions

$n=2$: $8^2 + 3^0 = 65$ which divides by 5

$\Rightarrow$ true for $n=2$

Assume true for $n=k$

$\Rightarrow 8^k + 3^{k-2} = 5m$ ① $m \in \mathbb{N}$

Consider $n = k+1$

$\Rightarrow 8^{k+1} + 3^{(k+1)-2}$

$= 8^{k+1} + 3^{k-1}$

$= 8 \cdot 8^k + 3 \cdot 3^{k-2}$

$= 8 \cdot 8^k + 8 \cdot 3^{k-2} - 5 \cdot 3^{k-2}$

$= 8(8^k + 3^{k-2}) - 5 \cdot 3^{k-2}$

$= 8 \cdot 5m - 5 \cdot 3^{k-2}$ by ①

$= 5(8m - 3^{k-2})$

$=$ multiple of 5 $\Rightarrow 8^{k+1} + 3^{k-1}$ divides by 5

$\Rightarrow$ true for $n=k \Rightarrow$ true for $n=k+1$

since true for $n=2 \Rightarrow$ true $\forall n, n \geq 2, n \in \mathbb{N}$

2011

Q16b – 5 marks

Define $I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,dx$ for $n \geq 1$.

(a) Use integration by parts to show that

$$I_n = \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx.$$ **3**

(b) Find the values of A and B for which

$$\frac{A}{(1+x^2)^n} + \frac{B}{(1+x^2)^{n+1}} = \frac{x^2}{(1+x^2)^{n+1}}$$

and hence show that

$$I_{n+1} = \frac{1}{n\times 2^{n+1}} + \left(\frac{2n-1}{2n}\right)I_n.$$ **5**

(c) Hence obtain the exact value of $\int_0^1 \frac{1}{(1+x^2)^3}\,dx$. **3**

Written Solutions

$$\frac{A}{(1+x^2)^n} + \frac{B}{(1+x^2)^{n+1}} = \frac{x^2}{(1+x^2)^{n+1}}$$

$$A(1+x^2) + B = x^2$$

$$A + Ax^2 + B = x^2$$

$$Ax^2 + A + B = x^2$$

∴ Comparing Coefficients:

x^2: $\underline{\underline{A = 1}}$

constant: $A + B = 0 \Rightarrow \underline{\underline{B = -1}}$

$$\frac{1}{(1+x^2)^n} - \frac{1}{(1+x^2)^{n+1}} = \frac{x^2}{(1+x^2)^{n+1}}$$

$$I_n = \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx$$

$$= \frac{1}{2^n} + 2n\int_0^1 \left(\frac{1}{(1+x^2)^n} - \frac{1}{(1+x^2)^{n+1}}\right)dx$$

$$= \frac{1}{2^n} + 2n(I_n - I_{n+1})$$

$$= \frac{1}{2^n} + 2nI_n - 2nI_{n+1}$$

$$2nI_{n+1} = \frac{1}{2^n} + 2nI_n - I_n \quad *$$

$$* \; I_{n+1} = \frac{1}{2n2^n} + \left(\frac{2n-1}{2n}\right)I_n$$

$$I_{n+1} = \frac{1}{n2^{n+1}} + \left(\frac{2n-1}{2n}\right)I_n$$

2010

Q8 – 6 marks

(*a*) Prove that the product of two odd integers is odd. **2**

(*b*) Let p be an odd integer. Use the result of (*a*) to prove by induction that p^n is odd for all positive integers n. **4**

Written Solutions

(a) let $z_1 = 2m+1$, $z_2 = 2n+1$ $\quad m, n \in \mathbb{Z}$

$$\begin{aligned} z_1 z_2 &= (2m+1)(2n+1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \\ &= \text{even} + 1 \\ &= \text{odd} \end{aligned}$$

$z_1 z_2 = \text{odd}$

(b) p odd $\Rightarrow p^n$ is odd $\quad n \in \mathbb{N}$

$n = 1 \Rightarrow p^1 = p = \text{odd}$ by defn

true for $n = 1$

Assume odd for $n = k$

$\Rightarrow p^k$ is odd ①

Consider $n = k+1$

$$\begin{aligned} \Rightarrow p^{k+1} &= p \cdot p^k \\ &= \text{odd} \times \text{odd} \quad \text{since ①} \\ &= \text{odd} \quad \text{true by (a)} \end{aligned}$$

$\Rightarrow$ true for $n = k \Rightarrow$ true for $n = k+1$

Since true for $n = 1$, by induction

true $\forall n \in \mathbb{N}$

2010

Q12 – 4 marks

Prove by contradiction that if x is an irrational number, then $2 + x$ is irrational. 4

Written Solutions

Assume x is rational

$\Rightarrow \quad x = \frac{a}{b} \qquad a, b \in \mathbb{Z}$

$\Rightarrow \quad 2 + x = 2 + \frac{a}{b}$

$2 + x = \frac{2b + a}{b}$

$=$ rational since $2b + a \in \mathbb{Z}$ by closure.

$\Rightarrow$ contradiction by 1

$\Rightarrow$ x is irrational.

OR Assume $2 + x$ is rational

$\Rightarrow \quad 2 + x = \frac{m}{n}$

$\Rightarrow \quad x = \frac{m}{n} - 2$

$= \frac{m - 2n}{n}$

Since $m - 2n$ and n are all integers, it follows that x is rational. This is a contradiction.

2009

Q4 – 5 marks

Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1}. \qquad \text{AIM} \qquad 1 - \frac{1}{(k+1)+1} \qquad \mathbf{5}$$

Written Solutions

$n=1$: LHS $\frac{1}{1(1+1)} = \frac{1}{2}$ RHS $1 - \frac{1}{1+1} = \frac{1}{2}$ LHS = RHS $\Rightarrow$ true for $n=1$

Assume true for $n=k \Rightarrow \displaystyle\sum_{r=1}^{k} \frac{1}{r(r+1)} = 1 - \frac{1}{k+1}$ ①

Consider $n=k+1$ $\displaystyle\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+1+1)}$

$$= 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)} \quad \text{by ①}$$

$$= 1 - \left(\frac{k+2-1}{(k+1)(k+2)} \right)$$

$$= 1 - \frac{k+1}{(k+1)(k+2)}$$

$$= 1 - \frac{1}{k+2}$$

$$= 1 - \frac{1}{(k+1)+1}$$

$\Rightarrow$ true for $n=k \Rightarrow$ true for $n=k+1$

$\Rightarrow$ since true for $n=1 \Rightarrow$ by induction true $\forall n$ $n \in \mathbb{N}$

2008

Q11 – 5 marks

For each of the following statements, decide whether it is true or false and prove your conclusion.

A For all natural numbers m, if m^2 is divisible by 4 then m is divisible by 4.

B The cube of any odd integer p plus the square of any even integer q is always odd. **5**

Written Solutions

A If m^2 is divisible by 4, m is divisible by 4.

By using a counterexample I will disprove this:

Let $m = 2$ $m^2 = 4$, which is divisible by 4

But if $m = 2$, $m =$ not divisible by 4

$\therefore$ A is false

B p is odd, let $p = 2k + 1$

q is even, let $q = 2m$

Then

$$p^3 + q^2$$
$$= (2k+1)^3 + (2m)^2$$
$$= 8k^3 + 12k^2 + 6k + 1 + 4m^2$$
$$= 2(4k^3 + 6k^2 + 3k + 2m^2) + 1$$
$$= 2N + 1 \Rightarrow \text{always odd}$$

Thus B is true

2007

Q12 – 5 marks

Prove by induction that for $a > 0$,

$$(1+a)^n \geq 1 + na \qquad 1 + (k+1)a$$

for all positive integers n. **5**

Written Solutions

$n = 1$: LHS $= (1+a)^1$ RHS $= 1 + 1.a$

$= 1 + a$ $= 1 + a$

LHS = RHS

$\Rightarrow$ true for $n = 1$

Assume true for $n = k$: $(1+a)^k \geq 1 + ka$ ①

Consider $n = k+1$

$$(1+a)^{k+1} = (1+a)\cdot(1+a)^k$$

$$\geq (1+a)(1+ka) = 1 + a + ka + ka^2$$

$$= 1 + (k+1)a \underbrace{+ka^2}_{\text{extra!!}}$$

$$\geq 1 + (k+1)a$$

since $ka^2 > 0$

$\Rightarrow$ required result for $n = k+1$

$\Rightarrow$ true for $n = k \Rightarrow$ true for $n = k+1$

$\Rightarrow$ since true for $n = 1 \Rightarrow$ by induction true $\forall n$, $n \in \mathbb{Z}^+$