Curve Sketching

Stationary Points and Points of Inflexion

The second derivative can be used to investigate the nature of stationary points on the curve y = f(x).

1. If $\frac{d^2y}{dx^2} > 0$ at a stationary point (SP), the SP is a **minimum turning point**.

2. If $\frac{d^2y}{dx^2} < 0$ at a SP, the SP is a **maximum turning point**.

3. If $\frac{d^2y}{dx^2} = 0$ at a SP, a **point of inflexion** occurs.

Example

Determine the coordinates and nature of the SP's on the curve $y = 2x^3 - 3x^2 - 12x$.

Coordinates of Stationary Points

$$\frac{dy}{dx} = 6x^2 - 6x - 12$$

$$\frac{dy}{dx} = 0 \quad @ \text{ SP's}$$

$6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$(x + 1)(x - 2) = 0$

So, $x = -1$ and $x = 2$

When $x = 2$, $y = -20$ i.e. $(2, -20)$

When $x = -1$, $y = 7$ i.e. $(-1, 7)$

Nature of Stationary Points

$$\frac{dy}{dx} = 6x^2 - 6x - 12$$

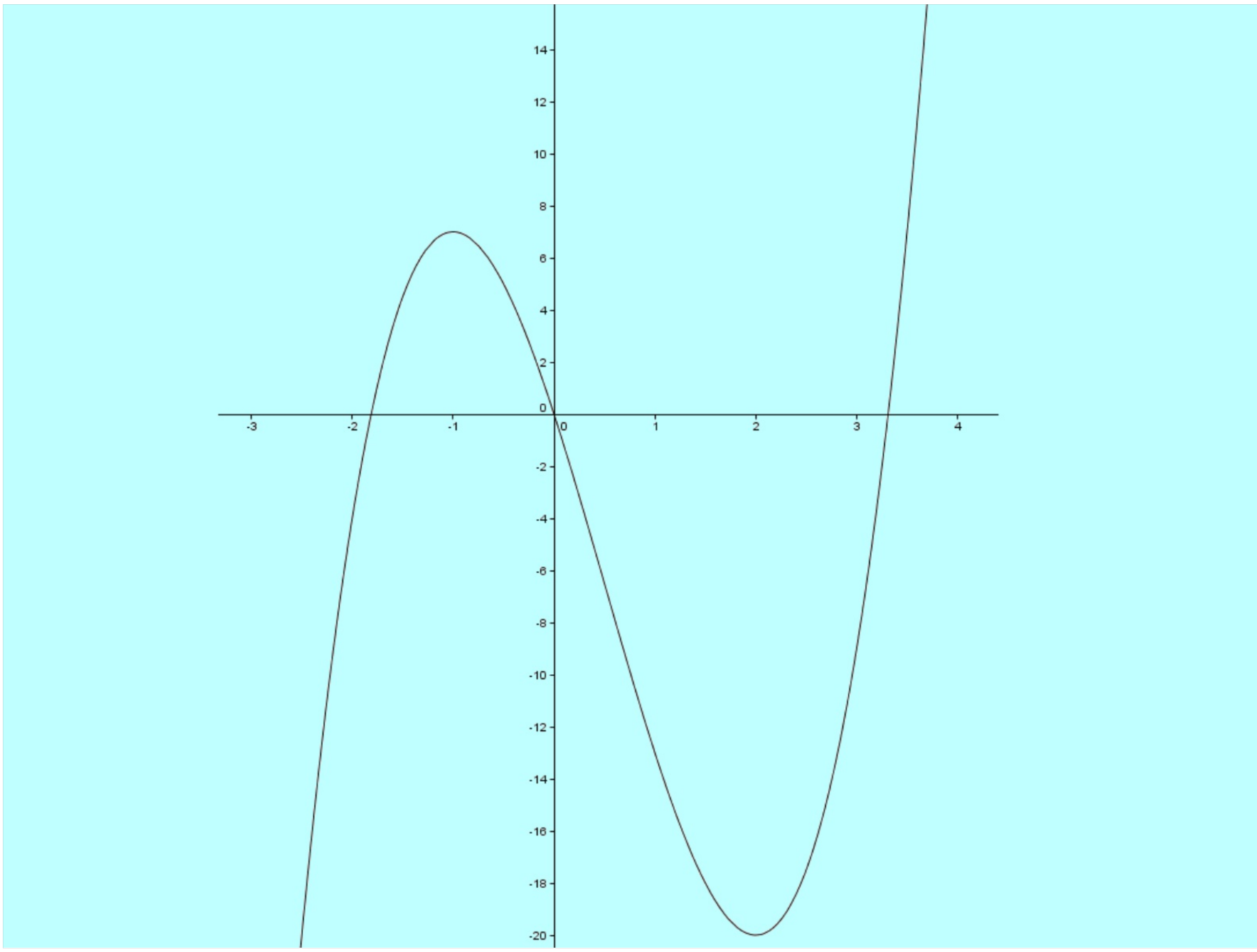
$$\frac{d^2y}{dx^2} = 12x - 6$$

When $x = 2$, $\frac{d^2y}{dx^2} = 12(2) - 6 = 18 > 0$

$\therefore$ $(2, -20)$ is a minimum TP

When $x = -1$, $\frac{d^2y}{dx^2} = 12(-1) - 6 = -18 < 0$

$\therefore$ $(-1, 7)$ is a maximum TP



Example

Find the coordinates of the points of inflexion on the curve $y = x^4 - 6x^2$.

$$\frac{dy}{dx} = 4x^3 - 12x$$

$$\frac{d^2y}{dx^2} = 12x^2 - 12$$

@ point of inflexion $\frac{d^2y}{dx^2} = 0$

$12x^2 - 12 = 0$

$12x^2 = 12$

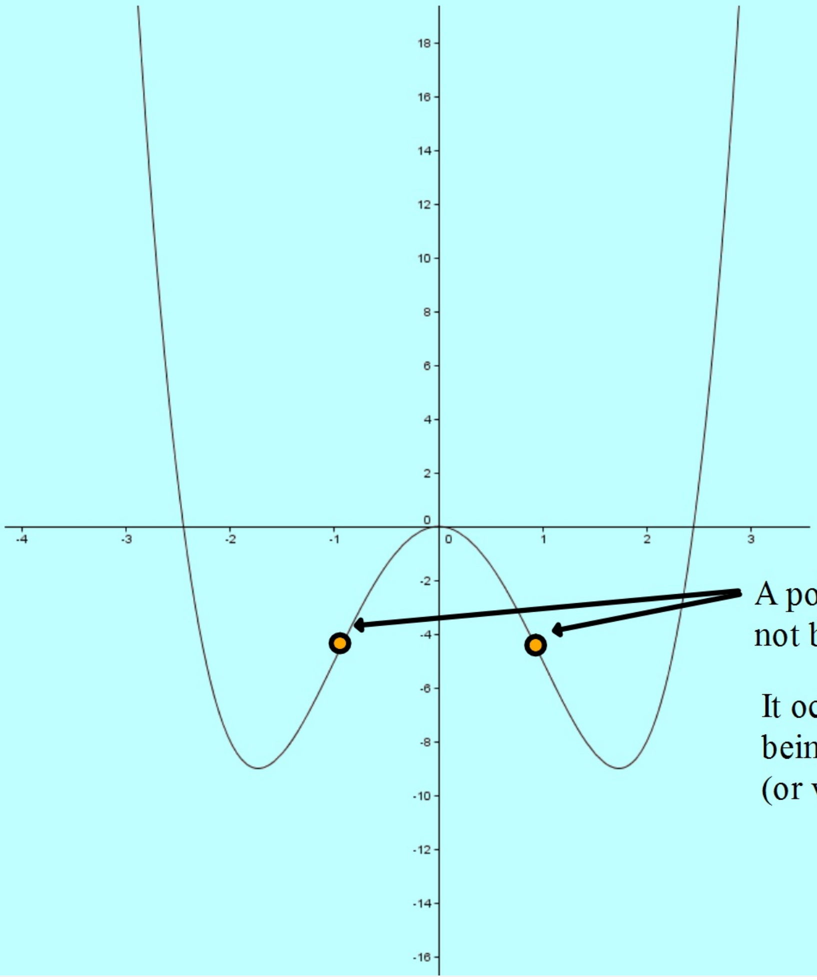
$x^2 = 1$

$x = \sqrt{1}$

$x = \pm 1$

When x = 1, y = -5 i.e. (1, -5)

When x = -1, y = -5 i.e. (-1, -5)



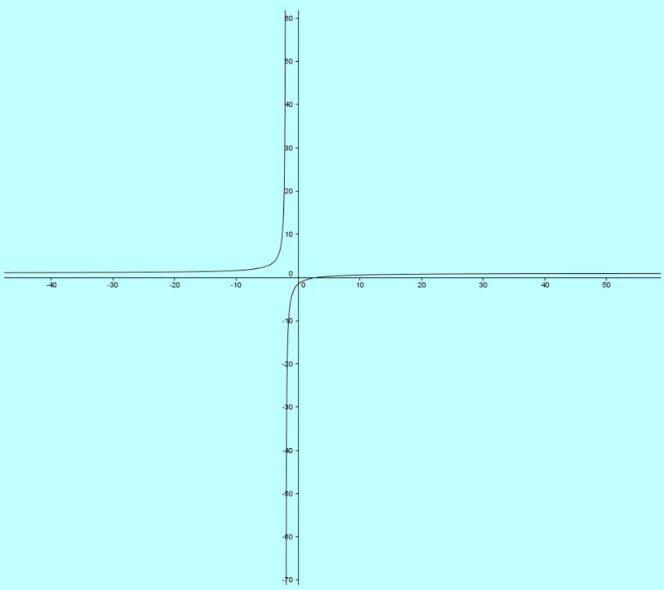
A point of inflexion need not be a SP.

It occurs when the curve changes from being concave up to concave down (or vice versa).

Exam Question

Show that there are no points of inflexion on the curve.

$$y = \frac{x-3}{x+2}, \ x \neq -2$$



$y = \frac{x-3}{x+2}$ where $u = x - 3$ $\quad u' = 1$

$v = x + 2$ $\quad v' = 1$

$$\frac{dy}{dx} = \frac{u'v - v'u}{v^2}$$

$$= \frac{x+2-(x-3)}{(x+2)^2}$$

$$= \frac{5}{(x+2)^2}$$

$$= \frac{5}{(x+2)^2}$$

$$= 5(x+2)^{-2}$$

$$\frac{d^2y}{dx^2} = -10(x+2)^{-3}$$

$$= -\frac{10}{(x+2)^3}$$

$\frac{d^2y}{dx^2} = 0$ @ points of inflexion $\therefore -\frac{10}{(x+2)^3} = 0$

Since this is not possible, there is a contradiction.

This means that there are no points of inflexion on the curve.