Vectors

2013

Q15 – 9 marks

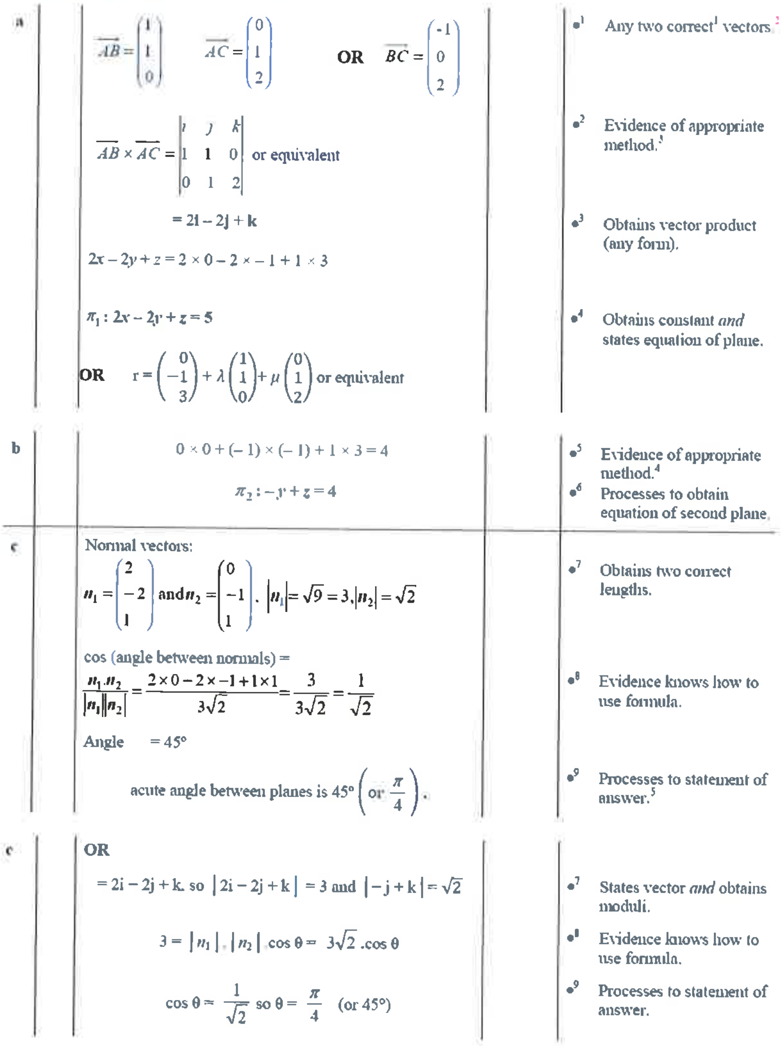
(*a*) Find an equation of the plane π_1, through the points $A(0, -1, 3)$, $B(1, 0, 3)$ and $C(0, 0, 5)$. **4**

(*b*) π_2 is the plane through A with normal in the direction $-\mathbf{j} + \mathbf{k}$.

Find an equation of the plane π_2. **2**

(*c*) Determine the acute angle between planes π_1 and π_2. **3**

Marking Instructions



a		$\overrightarrow{AB} = \begin{pmatrix}1\\1\\0\end{pmatrix}$ $\quad \overrightarrow{AC} = \begin{pmatrix}0\\1\\2\end{pmatrix}$ **OR** $\overrightarrow{BC} = \begin{pmatrix}-1\\0\\2\end{pmatrix}$		$\bullet^1$ Any two correct[1] vectors[2]
		$\overline{AB} \times \overline{AC} = \begin{vmatrix} i & j & k\\ 1 & 1 & 0\\ 0 & 1 & 2\end{vmatrix}$ or equivalent		$\bullet^2$ Evidence of appropriate method.[3]
		$= 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ $2x - 2y + z = 2 \times 0 - 2 \times -1 + 1 \times 3$		$\bullet^3$ Obtains vector product (any form).
		$\pi_1 : 2x - 2y + z = 5$ **OR** $\mathbf{r} = \begin{pmatrix}0\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}0\\1\\2\end{pmatrix}$ or equivalent		$\bullet^4$ Obtains constant *and* states equation of plane.
b		$0 \times 0 + (-1) \times (-1) + 1 \times 3 = 4$		$\bullet^5$ Evidence of appropriate method.[4]
		$\pi_2 : -y + z = 4$		$\bullet^6$ Processes to obtain equation of second plane.
c		Normal vectors: $n_1 = \begin{pmatrix}2\\-2\\1\end{pmatrix}$ and $n_2 = \begin{pmatrix}0\\-1\\1\end{pmatrix}$. $\lvert n_1\rvert = \sqrt{9} = 3, \lvert n_2\rvert = \sqrt{2}$		$\bullet^7$ Obtains two correct lengths.
		cos (angle between normals) = $\frac{n_1.n_2}{\lvert n_1\rvert\lvert n_2\rvert} = \frac{2\times0-2\times-1+1\times1}{3\sqrt{2}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$ Angle $= 45°$		$\bullet^8$ Evidence knows how to use formula.
		acute angle between planes is $45°\left(\text{or } \frac{\pi}{4}\right)$.		$\bullet^9$ Processes to statement of answer.[5]
c		**OR** $= 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. so $\lvert 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}\rvert = 3$ and $\lvert -\mathbf{j} + \mathbf{k}\rvert = \sqrt{2}$		$\bullet^7$ States vector *and* obtains moduli.
		$3 = \lvert n_1\rvert . \lvert n_2\rvert .\cos\theta = 3\sqrt{2}.\cos\theta$		$\bullet^8$ Evidence knows how to use formula.
		$\cos\theta = \frac{1}{\sqrt{2}}$ so $\theta = \frac{\pi}{4}$ (or 45°)		$\bullet^9$ Processes to statement of answer.

2012

Q5 – 5 marks

Obtain an equation for the plane passing through the points $P\,(-2, 1, -1)$, $Q\,(1, 2, 3)$ and $R\,(3, 0, 1)$. 5

Marking Instructions

Method 1		
$\overrightarrow{PQ} = 3\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and $\overrightarrow{QR} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$	1	$\overrightarrow{PR}$ could be used
A normal to the plane:		
$\overrightarrow{PQ} \times \overrightarrow{QR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 4 \\ 2 & -2 & -2 \end{vmatrix}$	1M	
$= \mathbf{i}\begin{vmatrix} 1 & 4 \\ -2 & -2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & 4 \\ 2 & -2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix}$		
$= 6\mathbf{i} + 14\mathbf{j} - 8\mathbf{k}$	1	
Hence the equation has the form:		
$6x + 14y - 8z = d$	1	
The plane passes through $P(-2, 1, -1)$ so		
$d = -12 + 14 + 8 = 10$		
which gives an equation $6x + 14y - 8z = 10$	1	
i.e. $3x + 7y - 4z = 5$		
Method 2		
A plane has an equation of the form		
$ax + by + cz = d$. Using the points P, Q, R we get		
$-2a + b - c = d$		
$a + 2b + 3c = d$	1M	
$3a + c = d$		
Using Gaussian elimination to solve these we have		or other valid method
$\begin{vmatrix} -2 & 1 & -1 & d \\ 1 & 2 & 3 & d \\ 3 & 0 & 1 & d \end{vmatrix} \Rightarrow \begin{vmatrix} -2 & 1 & -1 & d \\ 0 & 5 & 5 & 3d \\ 0 & 6 & 8 & 2d \end{vmatrix}$	1	
$\Rightarrow \begin{vmatrix} -2 & 1 & -1 & d \\ 0 & 5 & 5 & 3d \\ 0 & 0 & 2 & -\frac{8}{5}d \end{vmatrix}$	1	
$\Rightarrow c = -\frac{4}{5}d, \quad b = \frac{7}{5}d, \quad a = \frac{3}{5}d$	1	
These give the equation		
$(\frac{3}{5}d)x + (\frac{7}{5}d)y + (-\frac{4}{5}d)z = d$		
i.e. $3x + 7y - 4z = 5$	1	

2011

Q15 – 10 marks

The lines L_1 and L_2 are given by the equations

$$\frac{x-1}{k} = \frac{y}{-1} = \frac{z+3}{1} \text{ and } \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2},$$

respectively.

Find:

(*a*) the value of k for which L_1 and L_2 intersect and the point of intersection; 6

(*b*) the acute angle between L_1 and L_2. 4

Marking Instructions

(a) In terms of a parameter s, L_1 is given by

$x = 1 + ks,\ y = -s,\ z = -3 + s$ 1

In terms of a parameter t, L_2 is given by

$x = 4 + t,\ y = -3 + t,\ z = -3 + 2t$ 1

Equating the y coordinates
and equating the z coordinates:

$$\left.\begin{aligned} -s &= -3 + t \\ -3 + s &= -3 + 2t \end{aligned}\right\}$$ 1

Adding these

$$-3 = -6 + 3t$$

$$\Rightarrow t = 1 \Rightarrow s = 2.$$ 1

From the x coordinates

$$1 + ks = 4 + t$$

Using the values of s and t

$$1 + 2k = 5 \Rightarrow k = 2$$ 1

The point of intersection is: $(5, -2, -1)$. 1

(b) L_1 has direction $2\mathbf{i} - \mathbf{j} + \mathbf{k}$.
L_2 has direction $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. 1 For both directions.

Let the angle between L_1 and L_2 be θ, then

$$\cos\theta = \frac{(2\mathbf{i} - \mathbf{j} + \mathbf{k}).(\mathbf{i} + \mathbf{j} + 2\mathbf{k})}{|2\mathbf{i} - \mathbf{j} + \mathbf{k}||\mathbf{i} + \mathbf{j} + 2\mathbf{k}|}$$ 1

$$= \frac{2 - 1 + 2}{\sqrt{6}\sqrt{6}} = \frac{3}{6} = \frac{1}{2}$$ 1

$$\theta = 60°$$ 1

The angle between L_1 and L_2 is 60°.

2010

Q6 – 4 marks

Given $\mathbf{u} = -2\mathbf{i} + 5\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{w} = -\mathbf{i} + \mathbf{j} + 4\mathbf{k}$.

Calculate $\mathbf{u}.(\mathbf{v} \times \mathbf{w})$. **4**

Marking Instructions

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ -1 & 1 & 4 \end{vmatrix}$$ 1M a valid approach

$$= \mathbf{i}\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & -1 \\ -1 & 4 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix}$$ 1

$$= 9\mathbf{i} - 11\mathbf{j} + 5\mathbf{k}$$ 1

$$\mathbf{u}.(\mathbf{v} \times \mathbf{w}) = (-2\mathbf{i} + 0\mathbf{j} + 5\mathbf{k}).(9\mathbf{i} - 11\mathbf{j} + 5\mathbf{k})$$

$$= -18 + 0 + 25$$

$$= 7.$$ 1

2009

Q16 – 11 marks

(*a*) Use Gaussian elimination to solve the following system of equations

$$\begin{aligned} x + y - z &= 6 \\ 2x - 3y + 2z &= 2 \\ -5x + 2y - 4z &= 1. \end{aligned}$$ 5

(*b*) Show that the line of intersection, L, of the planes $x + y - z = 6$ and $2x - 3y + 2z = 2$ has parametric equations

$$\begin{aligned} x &= \lambda \\ y &= 4\lambda - 14 \\ z &= 5\lambda - 20. \end{aligned}$$ 2

(*c*) Find the acute angle between line L and the plane $-5x + 2y - 4z = 1$. 4

Marking Instructions

(a)

$$\begin{aligned} x + y - z &= 6 \\ 2x - 3y + 2z &= 2 \\ -5x + 2y - 4z &= 1 \end{aligned}$$

$$\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2 & -3 & 2 & 2 \\ -5 & 2 & -4 & 1 \end{array} \Rightarrow \begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & -5 & 4 & -10 \\ 0 & 7 & -9 & 31 \end{array} \Rightarrow \begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & -5 & 4 & -10 \\ 0 & 0 & -\frac{17}{5} & 17 \end{array}$$

1,1,1

$$z = 17 \div \left(\frac{-17}{5}\right) = -5$$ **1**

$$-5y - 20 = -10 \Rightarrow y = -2$$

$$x - 2 + 5 = 6 \Rightarrow x = 3$$ **1**

(b) Let $x = \lambda$.

Method 1

In first plane: $x + y - z = 6$.

$\lambda + (4\lambda - 14) - (5\lambda - 20) = 5\lambda - 5\lambda + 6 = 6.$ **1**

In the second plane:

$2x - 3y + 2z = 2\lambda - 3(4\lambda - 14) + 2(5\lambda - 20) = 5\lambda - 5\lambda + 2 = 2.$ **1**

Method 2

$$\begin{aligned} y - z = 6 - \lambda &\Rightarrow y = 6 + z - \lambda \\ -3y + 2z &= 2 - 2\lambda \\ (-18 - 3z + 3\lambda) + 2z &= 2 - 2\lambda \\ -z = 20 - 5\lambda &\Rightarrow z = 5\lambda - 20 \\ \text{and } y &= 4\lambda - 14 \end{aligned}$$

1 (after $-3y + 2z = 2 - 2\lambda$), **1** (after $z = 5\lambda - 20$)

Method 2

$$\begin{array}{lll} x + y - z = 6 & & (1) \\ 2x - 3y + 2z = 2 & & (2) \\ 5x \quad\;\; - z = 20 & & (2) + 3(1) \\ 4x - y \qquad = 14 & & (2) + 2(1) \end{array}$$ **1**

$$\begin{aligned} y &= 4x - 14 \\ z &= 5x - 20 \end{aligned}$$

$$x = \lambda, y = 4\lambda - 14, z = 5\lambda - 20$$ **1**

(c) Direction of L is $\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$, direction of normal to the plane is $-5\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$. Letting θ be the angle between these then

$$\cos\theta = \frac{-5 + 8 - 20}{\sqrt{42}\sqrt{45}}$$ **1M,1**

$$= \frac{-17}{3\sqrt{210}}$$

This gives a value of 113.0° which leads to the angle 113.0° − 90° = 23.0°. **1,1**

2008

Q14 – 10 marks

(*a*) Find an equation of the plane π_1 through the points $A(1, 1, 1)$, $B(2, -1, 1)$ and $C(0, 3, 3)$. **3**

(*b*) The plane π_2 has equation $x + 3y - z = 2$.

Given that the point $(0, a, b)$ lies on both the planes π_1 and π_2, find the values of a and b. Hence find an equation of the line of intersection of the planes π_1 and π_2. **4**

(*c*) Find the size of the acute angle between the planes π_1 and π_2. **3**

Marking Instructions

(a)

$$\overrightarrow{AB} = \mathbf{i} - 2\mathbf{j} \qquad \overrightarrow{AC} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$ 1

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 0 \\ -1 & 2 & 2 \end{vmatrix} = (-4 - 0)\mathbf{i} - (2 - 0)\mathbf{j} + (2 - 2)\mathbf{k}$$ 1

$$= -4\mathbf{i} - 2\mathbf{j}$$

Equation is

$$-4x - 2y = k$$

$$= -4(1) - 2(1) = -6$$ 1

$$\text{i.e. } -2x - y = -3$$

$$2x + y = 3$$

(b) In π_1: $2 \times 0 + a = 3 \Rightarrow a = 3.$ 1

In π_2: $0 + 3a - b = 2 \Rightarrow b = 3a - 2 = 7.$ 1

Hence the point of intersection is $(0, 3, 7)$.

Line of intersection: direction from

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -2 & 0 \\ 1 & 3 & -1 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j} - 10\mathbf{k}$$ 1

$$x = 0 + 2t;\ y = 3 - 4t;\ z = 7 - 10t$$ 1

There are many valid variations on this (including symmetric form) and these were marked on their merits.

(c) Let the angle be θ, then

$$\cos\theta = \left|\frac{(-4\mathbf{i} - 2\mathbf{j}).(\mathbf{i} + 3\mathbf{j} - \mathbf{k})}{\sqrt{4^2 + 2^2}\sqrt{1^2 + 3^2 + 1^2}}\right| = \left|\frac{-4 - 6}{\sqrt{20 \times 11}}\right| = \frac{5}{\sqrt{55}}$$ 1M, 1

or

$$\sin\theta = \left|\frac{(-4\mathbf{i} - 2\mathbf{j}) \times (\mathbf{i} + 3\mathbf{j} - \mathbf{k})}{\sqrt{4^2 + 2^2}\sqrt{1^2 + 3^2 + 1^2}}\right|$$ 1M

$$= \frac{\sqrt{2^2 + 4^2 + 10^2}}{\sqrt{20}\sqrt{11}} = \sqrt{\frac{120}{20 \times 11}} = \sqrt{\frac{6}{11}}$$ 1

Hence $\theta \approx 47.6°$. 1

2007

Q15 – 10 marks

Lines L_1 and L_2 are given by the parametric equations

$$L_1 : x = 2 + s,\ y = -s,\ z = 2 - s \qquad L_2 : x = -1 - 2t,\ y = t,\ z = 2 + 3t.$$

(*a*) Show that L_1 and L_2 do not intersect. **3**

(*b*) The line L_3 passes through the point $P(1,\ 1,\ 3)$ and its direction is perpendicular to the directions of both L_1 and L_2. Obtain parametric equations for L_3. **3**

(*c*) Find the coordinates of the point Q where L_3 and L_2 intersect and verify that P lies on L_1. **3**

(*d*) PQ is the shortest distance between the lines L_1 and L_2. Calculate PQ. **1**

Marking Instructions

(a) Equating the x-coordinates: $2 + s = -1 - 2t \Rightarrow s + 2t = -3$ (1)
Equating the y-coordinates: $-s = t \Rightarrow s = -t$ **1**
Substituting in (1): $-t + 2t = -3 \Rightarrow t = -3 \Rightarrow s = 3.$ **1**
Putting $s = 3$ in L_1 gives $(5, -3, -1)$ and $t = -3$ in L_2. $(5, -3, -7)$.
As the z coordinates differ, L_1 and L_2 do not intersect. **1**

(b) Directions of L_1 and L_2 are: $\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$. The vector product of these gives the direction of L_3.

$$(\mathbf{i}-\mathbf{j}-\mathbf{k})\times(-2\mathbf{i}+\mathbf{j}+3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ -2 & 1 & 3 \end{vmatrix} = -2\mathbf{i}-\mathbf{j}-\mathbf{k} \qquad \mathbf{1M,1}$$

Equation of L_3:

$$\begin{aligned} \mathbf{r} &= \mathbf{i} + \mathbf{j} + 3\mathbf{k} + (-2\mathbf{i} - \mathbf{j} - \mathbf{k})u \\ &= (1 - 2u)\mathbf{i} + (1 - u)\mathbf{j} + (3 - u)\mathbf{k} \end{aligned}$$

Hence L_3 is given by $x = 1 - 2u,\ y = 1 - u,\ z = 3 - u.$ **1**

(c) Solving the x and y coordinates of L_3 and L_2:

$$-1 - 2t = 1 - 2u \text{ and } t = 1 - u$$
$$\Rightarrow -1 = 3 - 4u \Rightarrow u = 1 \text{ and } t = 0 \qquad \mathbf{1}$$

The point of intersection, Q, is $(-1,\ 0,\ 2)$ since $2 + 3t = 2$ and $3 - u = 2.$ **1**
L_1 is $x = 2 + s, y = -s, z = 2 - s$. When $x = 1, s = -1$ and hence $y = 1$ and $z = 3$, i.e. P lies on L_1. **1**

(d) $PQ = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}.$ **1**