**Proofs**

**Example 1**

This is a simple, *direct proof* that the sum of two integers is itself an even number.

**Proof:** Consider two even integers $x$ and $y$.

Since they are even, they can be written as $x = 2a$ and $y = 2b$ respectively for integers $a$ and $b$.

Then the sum $x + y = 2a + 2b = 2(a + b)$

From this, it is clear that $x + y = \text{has 2 as a factor and therefore is even, so the sum of any two even integers is even.}$

**Example 2**

Given that $r$ and $s$ are rational numbers, show that $r + s$ is rational.

**Proof:** Since $r$ and $s$ are rational, we can write:

$$r = \frac{p}{q} \quad \text{and} \quad s = \frac{m}{n} \quad m, n, p, q \in \mathbb{Z}$$

Then

$$r + s = \frac{p}{q} + \frac{m}{n}$$

$$= \frac{pn + qm}{qn}$$

Since $m, n, p, q$ are integers, $pn + qm$ and $qn$ are also integers.

Thus, by the calculation above, $r + s$ is the quotient of two integers, and is therefore a rational number.

**Example 3**

Here's an example of a proof that is really just a calculation.

Given the trig identity $\sin(x + y) = \sin x \cos y + \cos x \sin y$, prove the identity: $\sin(2x) = 2\sin x \cos x$

**Proof:**

$$\sin(2x) = \sin(x + x)$$

$$= \sin x \cos x + \cos x \sin x$$

$$= 2\sin x \cos x$$
2013 Q12

Let \( n \) be a natural number.

For each of the following statements, decide whether it is true or false. If true, give a proof; if false, give a counterexample.

A If \( n \) is a multiple of 9 then so is \( n^2 \).

B If \( n^2 \) is a multiple of 9 then so is \( n \).

2010 Q8

(a) Prove that the product of two odd integers is odd.

(b) Let \( p \) be an odd integer. Use the result of (a) to prove by induction that \( p^n \) is odd for all positive integers \( n \).

2008 Q11

For each of the following statements, decide whether it is true or false and prove your conclusion.

A For all natural numbers \( m \), if \( m^3 \) is divisible by 4 then \( m \) is divisible by 4.

B The cube of any odd integer \( p \) plus the square of any even integer \( q \) is always odd.

### Solutions

#### 2013 Q12

**A**

Suppose \( n^2 \) is a multiple of 9.

Then \( n^2 = 9k \) for some integer \( k \).

Hence \( n = 3k \) for some integer \( k \).

**B**

Suppose \( n^2 \) is a multiple of 9.

Then \( n^2 = 9k \) for some integer \( k \).

Hence \( n = 3k \) for some integer \( k \).

### 2010 Q8

(a) Case 1: \( p = 2m + 1 \) and \( q = 2n + 1 \) where \( m \) and \( n \) are integers.

Then
\[
(pq)^2 = (2m+1)(2n+1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1
\]

which is odd.

(b) Let \( p = 2m + 1 \) and \( q = 2n + 1 \) where \( m \) and \( n \) are integers.

Then
\[
(2m+1)^2 = 4m^2 + 4m + 1
\]

which is odd.

### 2008 Q11

(a) Counterexample: \( m = 2 \).

(b) Let the numbers be \( 2m + 1 \) and \( 2n + 3 \).

Then
\[
(2m+1)^3 + (2n+3)^2 = 8m^3 + 6m^2 + 4m^2 + 6n^2 + 1 + 4n^2
\]

which is odd.

Proof:

Prove algebraically that either the cube of an odd number is odd or the square of an even number is even.

Odd cubed is odd and even squared is even.

So the sum of them is odd.