Expressing Improper, Rational Functions as Sums of Partial Fractions

Consider $\frac{8000}{46}$, an improper fraction. This can be rewritten as proper fraction by dividing out.

<u>BY LONG DIVISION</u>

$$
\begin{array}{rrrrrr}
 & & 0 & 1 & 7 & 3 \\
\cline{3-6}
4 & 6 & \multicolumn{1}{|r}{8} & 0 & 0 & 0 \\
 & - & 0 & \downarrow & & \\
\cline{3-3}
 & & 8 & 0 & & \\
 & - & 4 & 6 & \downarrow & \\
\cline{3-4}
 & & 3 & 4 & 0 & \\
 & - & 3 & 2 & 2 & \downarrow \\
\cline{3-5}
 & & & 1 & 8 & 0 \\
 & & - & 1 & 3 & 8 \\
\cline{4-6}
 & & & & 4 & 2
\end{array}
$$

Hence $\frac{8000}{46} = 173 + \frac{42}{46} = 173\frac{42}{46}$

We can apply the same number theory (method) to improper rational functions.

	• Divide the first term of the numerator by the first term of the denominator, and put that in the answer. • Multiply the denominator by that answer, put that below the numerator • Subtract to create a new polynomial
	Repeat, using the new polynomial

Example 1

Rewrite $\frac{x^3+2x^2-5}{x+3}$ as a proper rational function.

By Synthetic Division (when divisor is of degree 1).

-3	1	2	0	-5
		-3	3	9
	1	-1	3	-14

Hence $\frac{x^3+2x^2-5}{x+3} = x^2 - x + 3 - \frac{14}{x+3}$

By Long Division

$$
\begin{array}{r|l}
 & \quad\quad x^2 \; - \; x \; + \; 3 \\
\hline
x + 3 & x^3 + 2x^2 + 0x - 5 \\
 & \underline{x^3 + 3x^2} \quad \downarrow \\
 & \quad\quad -x^2 + 0x \\
 & \quad\quad \underline{-x^2 - 3x} \quad \downarrow \\
 & \quad\quad\quad\quad 3x - 5 \\
 & \quad\quad\quad\quad \underline{3x + 9} \\
 & \quad\quad\quad\quad\quad -14
\end{array}
$$

As before, $\dfrac{x^3+2x^2-5}{x+3} = x^2 - x + 3 - \dfrac{14}{x+3}$

NB/ If the degree of the numerator ≥ degree of denominator, then algebraic division must be used.

Example 2

Find partial fractions for $\dfrac{x^3+4x^2-x+2}{x^2+x}$.

$$
\begin{array}{rrrrrrrrr}
 & & & & & x & + & 3 \\ \cline{2-8}
x^2 + x & \big) \, x^3 & + & 4x^2 & - & x & + & 2 \\
 & x^3 & + & x^2 & & \downarrow & & \\ \cline{2-4}
 & & & 3x^2 & - & x & & \\
 & & & 3x^2 & + & 3x & & \downarrow \\ \cline{4-6}
 & & & & & -4x & + & 2
\end{array}
$$

$$\frac{x^3+4x^2-x+2}{x^2+2} = x + 3 - \frac{4x+2}{x^2+x}$$

$$= x + 3 - \frac{4x+2}{x(x+1)}$$

Now let $-\frac{4x+2}{x^2+x} = \frac{A}{x} + \frac{B}{x+1}$

× both sides by $x(x+1)$

$4x+2 = A(x+1) + Bx$

Let $x = -1$ $B = -6$

Let $x = 0$ $A = 2$

Hence $-\frac{4x+2}{x(x+1)} = \frac{2}{x} - \frac{6}{x+1}$

and so $\frac{x^3+4x^2-x+2}{x^2+x} = x + 3 + \frac{2}{x} - \frac{6}{x+1}$