Binomial

2013

Q1 – 4 marks

Write down the binomial expansion of $\left(3x - \frac{2}{x^2}\right)^4$ and simplify your answer.

Written Solutions

$$\left(3x - \frac{2}{x^2}\right)^4$$

$$= \binom{4}{0}(3x)^4(-2x^{-2})^0 + \binom{4}{1}(3x)^3(-2x^{-2})^1 + \binom{4}{2}(3x)^2(-2x^{-2})^2 + \binom{4}{3}(3x)^1(-2x^{-2})^3 + \binom{4}{4}(3x)^0(-2x^{-2})^4$$

$$= (1)(81x^4)(1) + (4)(27x^3)(-2x^{-2}) + (6)(9x^2)(4x^{-4}) + (4)(3x)(-8x^{-6}) + (1)(1)(16x^{-8})$$

$$= \underline{\underline{81x^4 - 216x + 216x^{-2} - 96x^{-5} + 16x^{-8}}}$$

2012

Q4 – 5 marks

Write down and simplify the general term in the expansion of $\left(2x - \frac{1}{x^2}\right)^9$. 3

Hence, or otherwise, obtain the term independent of x. 2

Written Solutions

General term $\binom{9}{r}(2x)^{9-r}\left(-\frac{1}{x^2}\right)^r$

$$= \binom{9}{r} \times 2^{9-r} \times x^{9-r} \times -\frac{1^r}{x^{2r}}$$

$$= \binom{9}{r} \times (-1)^r \, 2^{9-r} \, x^{9-3r}$$

'simplify 'x' terms'

The term independent of x occurs when:

$9 - 3r = 0$ i.e. $r = 3$

The term is: $\frac{9!}{6!\,3!}(-1)^3\,2^6$

$$= \underline{\underline{-5376}}$$

2011

Q2 – 3 marks

Use the binomial theorem to expand $\left(\frac{1}{2}x-3\right)^4$ and simplify your answer.

Written Solutions

$$\left(\frac{1}{2}x-3\right)^4$$

$$= \binom{4}{0}\left(\tfrac{1}{2}x\right)^4(-3)^0 + \binom{4}{1}\left(\tfrac{1}{2}x\right)^3(-3)^1 + \binom{4}{2}\left(\tfrac{1}{2}x\right)^2(-3)^2 + \binom{4}{3}\left(\tfrac{1}{2}x\right)^1(-3)^3 + \binom{4}{4}\left(\tfrac{1}{2}x\right)^0(-3)^4$$

$$= (1)\left(\tfrac{1}{16}x^4\right)(1) + (4)\left(\tfrac{1}{8}x^3\right)(-3) + 6\left(\tfrac{1}{4}x^2\right)(9) + 4\left(\tfrac{1}{2}x\right)(-27) + 1(1)(81)$$

$$= \underline{\underline{\frac{x^4}{16} - \frac{3x^3}{2} + \frac{27x^2}{2} - 54x + 81}}$$

2010

Q5 – 4 marks

Show that

$$\binom{n+1}{3} - \binom{n}{3} = \binom{n}{2}$$

where the integer n is greater than or equal to 3.

Written Solutions

$$\binom{n+1}{3} - \binom{n}{3}$$

$$= \frac{(n+1)!}{3!\,(n-2)!} - \frac{n!}{3!\,(n-3)!}$$

$$= \frac{(n+1)!\,(n-3)!}{3!\,(n-2)!\,(n-3)!} - \frac{n!\,(n-2)!}{3!\,(n-2)!\,(n-3)!}$$

$$= \frac{(n+1)!\,(n-3)! - n!\,(n-2)!}{3!\,(n-2)!\,(n-3)!}$$

2009

Q8 – 3 marks

(*a*) Write down the binomial expansion of $(1 + x)^5$.

(*b*) Hence show that $0{\cdot}9^5$ is $0{\cdot}59049$.

Written Solutions

(a) $(1 + x)^5$

$= \underline{\underline{1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5}}$

(b) let $x = -0{\cdot}1$

$0{\cdot}9^5 = (1 + (-0{\cdot}1))^5$

$= 1 - 0{\cdot}5 + 10(-0{\cdot}1)^2 + 10(-0{\cdot}1)^3 + 5(0{\cdot}1)^4 + (-0{\cdot}1)^5$

$= 1 - 0{\cdot}5 + 0{\cdot}1 - 0{\cdot}01 + 0{\cdot}0005 - 0{\cdot}00001$

$= \underline{\underline{0{\cdot}59049}}$ as req^{d}

2008

Q8 – 5 marks

Write down and simplify the general term in the expansion of $\left(x^2 + \frac{1}{x}\right)^{10}$. 3

Hence, or otherwise, obtain the term in x^{14}. 2

Written Solutions

General term $\binom{10}{r}(x^2)^{10-r}\,(x^{-1})^r$

$$= \binom{10}{r} x^{20-2r}\, x^{-r}$$

$$= \binom{10}{r} x^{20-3r}$$

We want the x^{14} term. Hence $20 - 3r = 14$

$\Rightarrow \quad r = 2$

Using $\binom{10}{r} x^{20-3r}$ the term is $\binom{10}{2} x^{20-3(2)}$

$$= \underline{\underline{45x^{14}}}$$

2007

Q1 – 4 marks

Express the binomial expansion of $\left(x-\frac{2}{x}\right)^4$ in the form $ax^4+bx^2+c+\frac{d}{x^2}+\frac{e}{x^4}$ for integers a, b, c, d and e.

Written Solutions

$$\left(x-\frac{2}{x}\right)^4$$

$$= \binom{4}{0}x^4(-2x^{-1})^0 + \binom{4}{1}x^3(-2x^{-1})^1 + \binom{4}{2}x^2(-2x^{-1})^2 + \binom{4}{3}x^1(-2x^{-1})^3 + \binom{4}{4}x^0(-2x^{-1})^4$$

$$= (1)x^4(1) + (4)x^3(-2x^{-1}) + (6)x^2(4x^{-2}) + (4)x(-8x^{-3}) + (1)(1)(16x^{-4})$$

$$= x^4 - 8x^2 + 24x^0 - 32x^{-2} + 16x^4$$

$$= x^4 - 8x^2 + 24 - \frac{32}{x^2} + \frac{16}{x^4}$$

2005

Q12 (a) – 3 marks *also involves "Complex Numbers" from Unit 3*

Let $z = \cos\theta + i\sin\theta$.

(*a*) Use the binomial expansion to express z^4 in the form $u + iv$, where u and v are expressions involving $\sin\theta$ and $\cos\theta$.

Written Solutions

$$z^4 = (\cos\theta + i\sin\theta)^4$$

$$= \binom{4}{0}\cos^4\theta\,(i\sin\theta)^0 + \binom{4}{1}\cos^3\theta\,(i\sin\theta)^1 + \binom{4}{2}\cos^2\theta\,(i\sin\theta)^2 + \binom{4}{3}\cos\theta\,(i\sin\theta)^3 + \binom{4}{4}(\cos\theta)^0(i\sin\theta)^4$$

$$= (1)\cos^4\theta(1) + (4)\cos^3\theta(i\sin\theta) + (6)\cos^2\theta(i^2\sin^2\theta) + (4)\cos\theta(i^3\sin^3\theta) + (1)(1)(i^4\sin^4\theta)$$

$$= \cos^4\theta + 6\cos^2\theta\, i^2\sin^2\theta + i^4\sin^4\theta + 4\cos^3\theta\, i\sin\theta + 4\cos\theta\, i^3\sin^3\theta$$

$$= \underline{\underline{(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + i(4\cos^3\theta\sin\theta + 4\cos\theta\sin^3\theta)}}$$

Complex numbers 'stuff' to help with last line

$i^2 = -1$

$i^3 =$

$i^4 = 1$

2004

Q2 – 3 marks

Obtain the binomial expansion of $(a^2 - 3)^4$.

Written Solutions

$$(a^2 - 3)^4$$

$$= (1)(a^2)^4(-3)^0 + (4)(a^2)^3(-3)^1 + (6)(a^2)^2(-3)^2 + (4)(a^2)(-3)^3 + (1)(a^2)^0(-3)^4$$

$$= \underline{\underline{a^8 - 12a^6 + 54a^4 - 108a^2 + 81}}$$

2003

A9 (a bit of) – 2 marks

Expand $(w + w^{-1})^4$ by the binomial theorem

Written Solutions

$$(w + w^{-1})^4$$

$$= (1) w^4 (w^{-1})^0 + (4) w^3 (w^{-1})^1 + (6) w^2 (w^{-1})^2 + (4) w^1 (w^{-1})^3 + (1) w^0 (w^{-1})^4$$

$$= \underline{\underline{w^4 + 4w^2 + 6 + 4w^{-2} + w^{-4}}}$$