# Complex Numbers

## 2013

### Q7 – 4 marks

Given that $z = 1-\sqrt{3}\,i$, write down $\bar{z}$ and express $\bar{z}^2$ in polar form. **4**

### Marking Instructions

| | |
|---|---|
| $\bar{z} = 1+\sqrt{3}\,i$ | $\bullet^1$ Correct statement of conjugate. |
| $\bar{z} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ | $\bullet^2$ One of $r$, $\theta$ correct.[1]<br>$\bullet^3$ Second correct and accurate substitution.[1] |
| $\bar{z}^2 = \left[2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right]^2 = 4\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$ | $\bullet^4$ Processes to answer.[4,5] |

**OR**

| | |
|---|---|
| $\bar{z}^2 = \left(1+\sqrt{3}\,i\right)^2 = 1+2\sqrt{3}\,i-3 = -2+2\sqrt{3}\,i$ | $\bullet^2$ Obtains $\bar{z}^2$ in Cartesian form. |
| $\bar{z}^2 = -2+2\sqrt{3}\,i = r(\cos\theta + i\sin\theta)$ | |
| $r = 4, \theta = \frac{2\pi}{3}$, $\bar{z}^2 = 4\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$ | $\bullet^3$ One of $r$, $\theta$ correct.<br>$\bullet^4$ Second correct and accurate substitution.[4] |

# 2013

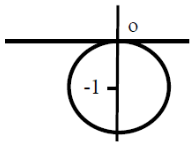
## Q10 – 5 marks

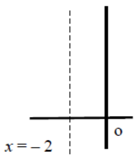
Describe the loci in the complex plane given by:

(a) $|z + i| = 1$; **2**

(b) $|z - 1| = |z + 5|$. **3**

## Marking Instructions

| | |
|---|---|
| Circle... | •[1] Observation that locus will be a circle.[8] |
| ...centre $(0, -1)$ [or $-i$], radius 1 | •[2] Identification of centre[1] (in either form) and radius.[8] |
| **OR** $z + i = x + iy + i = x + i(y + 1)$ | |
| $\lvert x + (y+1)i \rvert^2 = 1$ | •[1] Correct expression for modulus in Cartesian form.[5] |
| $x^2 + (y+1)^2 = 1$ | |
| Circle centre $(0, -1)$, radius 1 | •[2] Statement that locus is a circle, centre[1] (in either form) and radius. |
| **OR**  | •[1] Sketch of a circle<br>•[2] Identification of centre and radius.[2,6] |

| | |
|---|---|
| Set of points equidistant from $(1, 0)$ and $(-5, 0)$ | •[3] Observation that equidistant from specific points. |
| Straight line… | •[4] Identifies form of locus. |
| $\ldots x = -2$ | •[5] Statement of equation.[3] |
| **OR** $\lvert z-1 \rvert^2 = \lvert z+5 \rvert^2$ | |
| $\lvert (x-1) + iy \rvert^2 = \lvert (x+5) + iy \rvert^2$ | •[3] Collects real and imaginary parts *and* equates moduli.[4,8] |
| $(x-1)^2 + y^2 = (x+5)^2 + y^2$ | |
| $-2x + 1 = 10x + 25$ | |
| $-24 = 12x$ | |
| $x = -2$ | •[4] Accurately processes to reach equation. |
| which is a **straight line** | •[5] Explicitly states form of locus.[3] |
| **OR**  | •[3] Sketch of axes with any straight line drawn.<br>•[4] Vertical line to left of $y$-axis.<br>•[5] Explicitly states equation OR identifies the point (-2, 0) as being on the line. |

## 2012

### Q3 – 6 marks

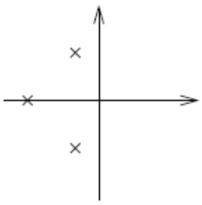
Given that $(-1 + 2i)$ is a root of the equation

$$z^3 + 5z^2 + 11z + 15 = 0,$$

obtain all the roots. **4**

Plot all the roots on an Argand diagram. **2**

### Marking Instructions

| | | |
|---|---|---|
| Since $w$ is a root, $\overline{w} = -1 - 2i$ is also a root. | **1** | for conjugate |
| The corresponding factors are $(z + 1 - 2i)$ and $(z + 1 + 2i)$ from which | | |
| $((z+1)-2i)((z+1)+2i) = (z+1)^2 + 4$ | | |
| $= z^2 + 2z + 5$ | **1** | |
| $z^3 + 5z^2 + 11z + 15 = (z^2 + 2z + 5)(z + 3)$ | **1** | evidence needed |
| The roots are $(-1 + 2i)$, $(-1 - 2i)$ and $-3$. | **1** | for stating roots together |
|  | **1** | for two correct points |
| | **1** | for third correct point |

## 2012

### Q16b – marks

(*a*) Prove by induction that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all integers $n \geq 1$. **6**

(*b*) Show that the real part of $\dfrac{\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)^{11}}{\left(\cos\frac{\pi}{36}+i\sin\frac{\pi}{36}\right)^{4}}$ is zero. **4**

### Marking Instructions

| | | |
|---|---|---|
| (a) For $n = 1$, the LHS $= \cos\theta + i\sin\theta$ and the RHS $= \cos\theta + i\sin\theta$. Hence the result is true for $n = 1$. | **1** | |
| Assume the result is true for $n = k$, i.e. $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$. | **1** | working with $n$ is penalised. |
| Now consider the case when $n = k + 1$: $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$ | **1** | |
| $= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$ | **1** | for applying the inductive hypothesis |
| $= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$ | **1** | multiplying and collecting |
| $= \cos(k+1)\theta + i\sin(k+1)\theta$ <br> Thus, if the result is true for $n = k$ the result is true for $n = k + 1$. <br> Since it is true for $n = 1$, the result is true for all $n \geqslant 1$. | **1** | |
| (b) $\dfrac{(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18})^{11}}{(\cos\frac{\pi}{36} + i\sin\frac{\pi}{36})^4} = \dfrac{\cos\frac{11\pi}{18} + i\sin\frac{11\pi}{18}}{\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}}$ | **1** | using result from above |
| $= \dfrac{\cos\frac{11\pi}{18} + i\sin\frac{11\pi}{18}}{\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}} \times \dfrac{\cos\frac{\pi}{9} - i\sin\frac{\pi}{9}}{\cos\frac{\pi}{9} - i\sin\frac{\pi}{9}}$ | **1** | |
| $= \dfrac{\cos\frac{11\pi}{18}\cos\frac{\pi}{9} + \sin\frac{11\pi}{18}\sin\frac{\pi}{9}}{\cos^2\frac{\pi}{9} + \sin^2\frac{\pi}{9}} + \text{imaginary term}$ | | |
| $= \cos\left(\dfrac{11\pi}{18} - \dfrac{\pi}{9}\right) + \text{imaginary term}$ | **1** | |
| $= \cos\dfrac{\pi}{2} + \text{imaginary term}$ <br> Thus the real part is zero as required. | **1** | or equivalent |

## 2011

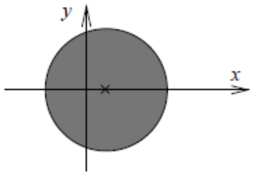
### Q10 – 5 marks

Identify the locus in the complex plane given by

$$|z - 1| = 3.$$

Show in a diagram the region given by $|z - 1| \leq 3$. **5**

### Marking Instructions

| | | |
|---|---|---|
| Let $z = x + iy$, so | | |
| $z - 1 = (x - 1) + iy$. | **1** | |
| $\lvert z - 1 \rvert^2 = (x - 1)^2 + y^2 = 9$. | **1** | |
| The locus is the circle with centre $(1,\ 0)$ and radius 3. | **1** | Can subsume the first two marks. |
|   | **1** | for circle |
| | **1** | for shading or other indication |

## 2010

### Q16 – 10 marks

Given $z = r(\cos\theta + i\sin\theta)$, use de Moivre's theorem to express $z^3$ in polar form. **1**

Hence obtain $\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3$ in the form $a + ib$. **2**

Hence, or otherwise, obtain the roots of the equation $z^3 = 8$ in Cartesian form. **4**

Denoting the roots of $z^3 = 8$ by $z_1$, $z_2$, $z_3$:

(*a*) state the value $z_1 + z_2 + z_3$;

(*b*) obtain the value of $z_1^6 + z_2^6 + z_3^6$. **3**

### Marking Instructions

| | | |
|---|---|---|
| $z^3 = r^3(\cos 3\theta + i\sin 3\theta)$ | **1** | |
| $(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})^3 = \cos 2\pi + i\sin 2\pi$ | **1** | necessary |
| $a = 1;\ b = 0$ | **1** | |
| *Method 1* | | |
| $r^3(\cos 3\theta + i\sin 3\theta) = 8$ | | |
| $r^3\cos 3\theta = 8$ and $r^3\sin 3\theta = 0$ | **1** | |
| $\Rightarrow r = 2;\ 3\theta = 0,\ 2\pi,\ 4\pi$ | **1** | |
| Roots are $2,\ 2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}),\ 2(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3})$. | **1** | |
| In cartesian form: $2,\ (-1 + i\sqrt{3}),\ (-1 - i\sqrt{3})$ | **1** | |
| *Method 2* | | |
| $z^3 - 8 = 0$ | **1** | |
| $(z - 2)(z^2 + 2z + 4) = 0$ | **1** | |
| $(z - 2)((z + 1)^2 + (\sqrt{3})^2) = 0$ | **1** | or by using quadratic formula |
| so the roots are: $2,\ (-1 + i\sqrt{3}),\ (-1 - i\sqrt{3})$ | **1** | |
| (a) $z_1 + z_2 + z_3 = 0$ | **1** | |
| (b) Since $z_1^3 = z_2^3 = z_3^3 = 8$ | **1** | |
| it follows that | | |
| $z_1^6 + z_2^6 + z_3^6 = (z_1^3)^2 + (z_2^3)^2 + (z_3^3)^2$ | | |
| $= 3 \times 64 = 192$ | **1** | |

## 2009

### Q6 – 6 marks

Express $z = \dfrac{(1+2i)^2}{7-i}$ in the form $a + ib$ where $a$ and $b$ are real numbers.

Show $z$ on an Argand diagram and evaluate $|z|$ and $\arg(z)$. **6**

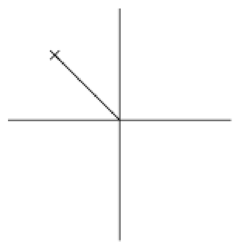
### Marking Instructions

$$\frac{(1+2i)^2}{7-i} = \frac{1+4i-4}{7-i} \qquad \mathbf{1}$$

$$= \frac{-3+4i}{7-i} \times \frac{7+i}{7+i} \qquad \mathbf{1}$$

$$= \frac{(-3+4i)(7+i)}{50}$$

$$= -\frac{1}{2} + \frac{1}{2}i \qquad \mathbf{1}$$



**1**

$$|z| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{2}\sqrt{2} \qquad \mathbf{1}$$

$$\arg z = \tan^{-1}\frac{\frac{1}{2}}{-\frac{1}{2}} = \tan^{-1}(-1) = \frac{3\pi}{4} \text{ (or } 135°\text{)}. \qquad \mathbf{1}$$

## 2008

### Q16 – 10 marks

Given $z = \cos\theta + i\sin\theta$, use de Moivre's theorem to write down an expression for $z^k$ in terms of $\theta$, where $k$ is a positive integer.

Hence show that $\dfrac{1}{z^k} = \cos k\theta - i\sin k\theta$. **3**

Deduce expressions for $\cos k\theta$ and $\sin k\theta$ in terms of $z$. **2**

Show that $\cos^2\theta\ \sin^2\theta = -\dfrac{1}{16}\left(z^2 - \dfrac{1}{z^2}\right)^2$. **3**

Hence show that $\cos^2\theta\ \sin^2\theta = a + b\cos 4\theta$, for suitable constants $a$ and $b$. **2**

## Marking Instructions

$z^k = \cos k\theta + i \sin k\theta,$ **1**

so $\frac{1}{z^k} = \frac{1}{\cos k\theta + i \sin k\theta} = \frac{\cos k\theta - i \sin k\theta}{\cos^2 k\theta + \sin^2 k\theta} = \cos k\theta - i \sin k\theta.$ **2E1**

Adding the expressions for $z^k$ and $\frac{1}{z^k}$ gives $z^k + \frac{1}{z^k} = 2 \cos k\theta$ so
$\cos k\theta = \frac{1}{2}(z^k + z^{-k}).$ **1**

Subtracting the expressions for $z^k$ and $\frac{1}{z^k}$ gives $z^k - \frac{1}{z^k} = 2i \sin k\theta$ so
$\sin k\theta = \frac{1}{2i}(z^k - z^{-k}).$ **1**

For $k = 1$

$$\cos^2\theta \sin^2\theta = (\cos\theta \sin\theta)^2$$ **1**

$$= \left(\frac{(z + \frac{1}{z})(z - \frac{1}{z})}{4i}\right)^2$$ **2E1**

$$= -\frac{1}{16}\left(z^2 - \frac{1}{z^2}\right)^2.$$

$$\left(z^2 - \frac{1}{z^2}\right)^2 = z^4 + \frac{1}{z^4} - 2 = 2\cos 4\theta - 2$$ **1**

$$\Rightarrow \cos^2\theta \sin^2\theta = \tfrac{1}{8} - \tfrac{1}{8}\cos 4\theta,$$ **1**

i.e. $a = \frac{1}{8}$ and $b = \frac{1}{8}$.

*OR*

A correct trigonometric proof that $\cos^2\theta \sin^2\theta = \frac{1}{8} - \frac{1}{8}\cos 4\theta$. **1**

# 2007

## Q3 – 4 marks

Show that $z = 3 + 3i$ is a root of the equation $z^3 - 18z + 108 = 0$ and obtain the remaining roots of the equation. **4**

## Marking Instructions

$(3 + 3i)^3 = 27 + 81i + 81i^2 + 27i^3 = -54 + 54i$. Thus

$$(3 + 3i)^3 - 18(3 + 3i) + 108 =$$

$$-54 + 54i - 54 - 54i + 108 = 0$$ **1**

Since $3 + 3i$ is a root, $3 - 3i$ is a root. **1**

These give a factor $\left(z - (3 + 3i)\right)\left(z - (3 - 3i)\right) = (z - 3)^2 + 9 = z^2 - 6z + 18$. **1**

$$z^3 - 18z + 108 = \left(z^2 - 6z + 18\right)(z + 6)$$

The remaining roots are $3 - 3i$ and $-6$. **1**

## 2007

### Q11 – 4 marks

Given that $|z-2| = |z+i|$, where $z = x + iy$, show that $ax + by + c = 0$ for suitable values of $a$, $b$ and $c$. **3**

Indicate on an Argand diagram the locus of complex numbers $z$ which satisfy $|z-2| = |z+i|$. **1**

### Marking Instructions

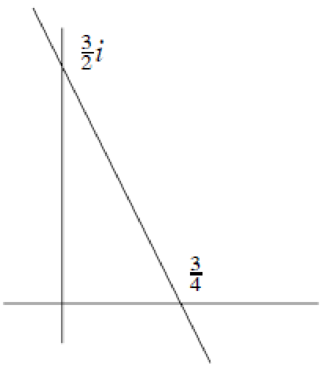
$$|z - 2| = |z + i|$$

$$|(x - 2) + iy| = |x + (y + 1)i| \quad \mathbf{1}$$

$$(x - 2)^2 + y^2 = x^2 + (y + 1)^2 \quad \mathbf{1}$$

$$-4x + 4 = 2y + 1$$

$$4x + 2y - 3 = 0 \quad \mathbf{1}$$




**1**