Further Differentiation

2013

Q11 – 6 marks

A curve has equation

$$x^2 + 4xy + y^2 + 11 = 0.$$

Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(-2, 3)$.

Marking Instructions

Expected Answer/s	Max Mark	Additional Guidance
A curve has equation $x^2 + 4xy + y^2 + 11 = 0$ **Find the values of** $\frac{dy}{dx}$ **and** $\frac{d^2y}{dx^2}$ **at the point (- 2, 3).**	6	
$2x + 4x\frac{dy}{dx} + 4y \ldots$		$\bullet^1$ Differentiates x^2 and first product.
$\ldots + 2y\frac{dy}{dx} = 0 \quad (\Delta)$		$\bullet^2$ Differentiates $y^2 + 11 = 0$ correctly.
$2(-2) + 4(-2)\frac{dy}{dx} + 4(3) + 2(3)\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \mathbf{4}$		$\bullet^3$ Evaluates $\frac{dy}{dx}$.[4]
OR $\frac{dy}{dx} = -\frac{2x+4y}{4x+2y} = -\frac{x+2y}{2x+y} \quad (\dagger) \quad \therefore \frac{dy}{dx} = \mathbf{4}$		$\bullet^3$ Evaluates $\frac{dy}{dx}$ after rearranging.
Differentiating (Δ): $2 + 4x\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4\frac{dy}{dx} \ldots$		$\bullet^4$ Differentiates first three terms of (Δ) correctly, including a product.[3]
$\ldots + 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0$		$\bullet^5$ Differentiates final product of (Δ) correctly.[3]
$\therefore 2 + 4(-2)\frac{d^2y}{dx^2} + 8(4) + 2(3)\frac{d^2y}{dx^2}\ 2(4)^2 = 0$		
$\therefore \frac{d^2y}{dx^2} = \mathbf{33}$		$\bullet^6$ Evaluates $\frac{d^2y}{dx^2}$.[1]
OR Differentiating $(\dagger)$:		$\bullet^4$ Evidence of valid application of quotient (or product) rule.
$\frac{d^2y}{dx^2} = -\frac{(2x+y)\left(1+2\frac{dy}{dx}\right) - (x+2y)\left(2+\frac{dy}{dx}\right)}{(2x+y)^2}$		$\bullet^5$ Differentiates correctly.
$\frac{d^2y}{dx^2} = -\frac{(2(-2)+3)(1+2(4)) - ((-2)+2(3))(2+4)}{(2(-2)+3)^2} = \mathbf{33}$		$\bullet^6$ Evaluates $\frac{d^2y}{dx^2}$.[1, 4]

2012

Q13 – 10 marks

A curve is defined parametrically, for all t, by the equations

$$x = 2t + \frac{1}{2}t^2, \qquad y = \frac{1}{3}t^3 - 3t.$$

Obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ as functions of t.

Find the values of t at which the curve has stationary points and determine their nature.

Show that the curve has exactly two points of inflexion.

Marking Instructions

		Marks awarded for
$x = 2t + \frac{1}{2}t^2 \quad \Rightarrow \quad \frac{dx}{dt} = 2 + t$	1	
$y = \frac{1}{3}t^3 - 3t \quad \Rightarrow \quad \frac{dy}{dt} = t^2 - 3$	1	
$\frac{dy}{dx} = \frac{t^2 - 3}{2 + t}$	1	
$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{2t(2+t) - (t^2 - 3)}{(2+t)^2} = \frac{t^2 + 4t + 3}{(2+t)^2}$	1	
$\frac{d^2y}{dx^2} = \frac{t^2 + 4t + 3}{(2+t)^2} \times \frac{1}{2+t} = \frac{t^2 + 4t + 3}{(2+t)^3}$	1	
Stationary points when $\frac{dy}{dx} = 0$, i.e. $t^2 - 3 = 0 \Rightarrow t = \pm\sqrt{3}$	1	
When $t = \sqrt{3}$, $\frac{d^2y}{dx^2} = \frac{3 + 4\sqrt{3} + 3}{(2 + \sqrt{3})^3} > 0$ which gives a minimum.	1	no marks for using a nature table
When $t = -\sqrt{3}$, $\frac{d^2y}{dx^2} = \frac{3 - 4\sqrt{3} + 3}{(2 - \sqrt{3})^3} < 0$ which gives a maximum.	1	no marks for using a nature table
At a point of inflexion, $\frac{d^2y}{dx^2} = 0$.	1	
In this case, that means $t^2 + 4t + 3 = (t + 1)(t + 3) = 0$ and this has exactly two roots.	1	need to show 2 values exist
Note that this is a slimmed-down version of the complete story of points of inflexion.		

2011

Q3a – 3 marks

Obtain $\dfrac{dy}{dx}$ when y is defined as a function of x by the equation

$$y + e^y = x^2.$$

Marking Instructions

(a) *Method 1*		
$y + e^y = x^2$		
$\dfrac{dy}{dx} + e^y \dfrac{dy}{dx} = 2x$	**1M** **1**	for applying implicit differentiation for accuracy
$\dfrac{dy}{dx}(1 + e^y) = 2x \Rightarrow \dfrac{dy}{dx} = \dfrac{2x}{(1 + e^y)}$	**1**	
Method 2		
$\ln(y + e^y) = 2\ln x$	**1**	
$\dfrac{(1 + e^y)\frac{dy}{dx}}{y + e^y} = \dfrac{2}{x}$	**1**	
$\dfrac{dy}{dx} = \dfrac{2(y + e^y)}{x(1 + e^y)}$	**1**	
Method 3		
$y + e^y = x^2 \Rightarrow e^y = x^2 - y \Rightarrow y = \ln(x^2 - y)$	**1**	
$\dfrac{dy}{dx} = \dfrac{2x - \frac{dy}{dx}}{x^2 - y}$	**1**	
$\frac{dy}{dx}(x^2 - y) = 2x - \frac{dy}{dx} \Rightarrow \frac{dy}{dx}(x^2 - y + 1) = 2x$		
$\dfrac{dy}{dx} = \dfrac{2x}{x^2 - y + 1}$	**1**	

2011

Q7 – 4 marks

A curve is defined by the equation $y = \dfrac{e^{\sin x}(2+x)^3}{\sqrt{1-x}}$ for $x < 1$.

Calculate the gradient of the curve when $x = 0$.

Marking Instructions

Method 1		
$y = \dfrac{e^{\sin x}(2+x)^3}{\sqrt{1-x}}$		
$\Rightarrow \ln y = \ln\left(e^{\sin x}(2+x)^3\right) - \ln\left(\sqrt{1-x}\right)$	**1M**	for use of logs
$= \sin x + 3\ln(2+x) - \frac{1}{2}\ln(1-x)$	**1**	for preparing to differentiate
$\Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = \cos x + \dfrac{3}{2+x} + \dfrac{1}{2(1-x)}$	**1**	
$\dfrac{dy}{dx} = y\left(\cos x + \dfrac{3}{2+x} + \dfrac{1}{2(1-x)}\right)$		
When $x = 0, y = 8 \Rightarrow$		
gradient $= 8\left(1 + \dfrac{3}{2} + \dfrac{1}{2}\right) = 24.$	**1**	for final value

Method 2		
$y = \dfrac{e^{\sin x}(2+x)^3}{\sqrt{1-x}} \Rightarrow \dfrac{dy}{dx} =$		
$\dfrac{\frac{d}{dx}\left(e^{\sin x}(2+x)^3\right)\sqrt{1-x} - e^{\sin x}(2+x)^3\left(-\frac{1}{2}\frac{1}{\sqrt{1-x}}\right)}{(1-x)}$	**M1**	for use of quotient rule
$= \dfrac{\left[\cos x\, e^{\sin x}(2+x)^3 + 3e^{\sin x}(2+x)^2\right](1-x)}{(1-x)^{3/2}}$	**M1**	
$+ \dfrac{e^{\sin x}(2+x)^3}{2(1-x)^{3/2}}$	**1**	
When $x = 0$,		
gradient $= \dfrac{(2^3 + 3\times 2^2)}{1} + \dfrac{2^3}{2} = 20 + 4 = 24$	**1**	
Method 3		
$y = \frac{e^{\sin x}(2+x)^3}{\sqrt{1-x}}$		
$y\sqrt{1-x} = e^{\sin x}(2+x)^3$	**1**	
$\sqrt{1-x}\frac{dy}{dx} - \frac{1}{2}y(1-x)^{-1/2}$		
$= \cos x\, e^{\sin x}(2+x)^3 + 3e^{\sin x}(2+x)^2$	**1,1**	
when $x = 0, y = \frac{e^0 2^3}{1} = 8$. This leads to		
$\frac{dy}{dx} = 24$	**1**	

2010

Q13 – 10 marks

Given $y = t^3 - \frac{5}{2}t^2$ and $x = \sqrt{t}$ for $t > 0$, use parametric differentiation to express $\frac{dy}{dx}$ in terms of t in simplified form.

Show that $\frac{d^2y}{dx^2} = at^2 + bt$, determining the values of the constants a and b.

Obtain an equation for the tangent to the curve which passes through the point of inflexion.

Marking Instructions

Working	Marks	Notes
$y = t^3 - \frac{5}{2}t^2 \Rightarrow \frac{dy}{dt} = 3t^2 - 5t$	1	
$x = \sqrt{t} = t^{1/2} \Rightarrow \frac{dx}{dt} = \frac{1}{2}t^{-1/2}$	1	
$\Rightarrow \frac{dy}{dx} = \frac{3t^2 - 5t}{\frac{1}{2}t^{-1/2}}$	1	
$= 6t^{5/2} - 10t^{3/2}$	1	for eliminating fractions
$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$	1M	
$= \frac{6 \times \frac{5}{2}t^{3/2} - 10 \times \frac{3}{2}t^{1/2}}{\frac{1}{2}t^{-1/2}}$	1	
$= 30t^2 - 30t$	1	
i.e. $a = 30,\ b = -30$		
At a point of inflexion, $\frac{d^2y}{dx^2} = 0 \Rightarrow t = 0$ or 1		
But $t > 0 \Rightarrow t = 1 \Rightarrow \frac{dy}{dx} = -4$	1	the value of the gradient
and the point of contact is $(1, -\frac{3}{2})$	1	
Hence the tangent is		
$y + \frac{3}{2} = -4(x - 1)$	1	
i.e. $2y + 8x = 5$		

2009

Q1b – 4 marks

(*a*) Given $f(x) = (x + 1)(x - 2)^3$, obtain the values of x for which $f'(x) = 0$.

(*b*) Calculate the gradient of the curve defined by $\frac{x^2}{y} + x = y - 5$ at the point $(3, -1)$.

Marking Instructions

(a)

$$f(x) = (x + 1)(x - 2)^3$$

$$f'(x) = (x - 2)^3 + 3(x + 1)(x - 2)^2 \qquad \mathbf{1}$$

$$= (x - 2)^2((x - 2) + 3(x + 1))$$

$$= (x - 2)^2(4x + 1) \qquad \mathbf{1}$$

$$= 0 \text{ when } x = 2 \text{ and when } x = -\frac{1}{4}. \qquad \mathbf{1}$$

(b) *Method 1*

$$\frac{x^2}{y} + x = y - 5 \Rightarrow x^2 + xy = y^2 - 5y \qquad \mathbf{1}$$

$$2x + x\frac{dy}{dx} + y = 2y\frac{dy}{dx} - 5\frac{dy}{dx} \qquad \mathbf{2E1}$$

$$6 + 3\frac{dy}{dx} - 1 = -2\frac{dy}{dx} - 5\frac{dy}{dx}$$

$$5 = -10\frac{dy}{dx} \Rightarrow \mathbf{\frac{dy}{dx} = \frac{-1}{2}} \qquad \mathbf{1}$$

Note: a candidate may obtain $\frac{dy}{dx} = \frac{2x+y}{2y-x-5}$ and then substitute.

Method 2

$$\frac{x^2}{y} + x = y - 5$$

$$\frac{2xy - x^2\frac{dy}{dx}}{y^2} + 1 = \frac{dy}{dx} \qquad \mathbf{2E1}$$

$$\frac{-6 - 9\frac{dy}{dx}}{1} + 1 = \frac{dy}{dx}$$

$$-6 - 9\frac{dy}{dx} + 1 = \frac{dy}{dx}$$

$$\mathbf{\frac{dy}{dx} = -\frac{1}{2}} \qquad \mathbf{2E1}$$

Mathod 3

$$\frac{x^2}{y} + x = y - 5 \Rightarrow x^2\left(\frac{1}{y}\right) + x = y - 5$$

$$2x\frac{1}{y} + x^2\left(-\frac{1}{y^2}\right)\frac{dy}{dx} + 1 = \frac{dy}{dx} \qquad \mathbf{2E1}$$

$$-6 - 9\frac{dy}{dx} + 1 = \frac{dy}{dx} \Rightarrow \mathbf{\frac{dy}{dx} = -\frac{1}{2}} \qquad \mathbf{2E1}$$

Note: a candidate may obtain $\frac{dy}{dx} = \frac{2xy+y^2}{y^2+x^2}$ (in 2 and 3) and then substitute.

2009

Q11 – 5 marks

The curve $y = x^{2x^2+1}$ is defined for $x > 0$. Obtain the values of y and $\frac{dy}{dx}$ at the point where $x = 1$.

Marking Instructions

When $x = 1, y = 1.$ **1**

$$y = x^{2x^2+1}$$

$$\Rightarrow \ln y = \ln\left(x^{2x^2+1}\right) \quad \mathbf{1}$$

$$= (2x^2 + 1)\ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x^2+1}{x} + 4x\ln x \quad \mathbf{1,1}$$

Hence, when $x = 1, y = 1$ and

$$\frac{dy}{dx} = 3 + 0 = 3. \quad \mathbf{1}$$

2008

Q2b – 3 marks

(*a*) Differentiate $f(x) = \cos^{-1}(3x)$ where $-\frac{1}{3} < x < \frac{1}{3}$.

(*b*) Given $x = 2\sec\theta$, $y = 3\sin\theta$, use parametric differentiation to find $\frac{dy}{dx}$ in terms of θ.

Marking Instructions

(a)

$$f(x) = \cos^{-1}(3x)$$

$$f'(x) = \frac{-1}{\sqrt{1-(3x)^2}} \times 3 \quad \mathbf{1,1}$$

$$= \frac{-3}{\sqrt{1-9x^2}}$$

(b)

$$\frac{dx}{d\theta} = 2\sec\theta\tan\theta, \qquad \frac{dy}{d\theta} = 3\cos\theta \quad \mathbf{1,1}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos\theta}{2\sec\theta\tan\theta} \quad \mathbf{1}$$

$$= \frac{3\cos^3\theta}{2\sin\theta}$$

2008

Q5 – 6 marks

A curve is defined by the equation $xy^2 + 3x^2y = 4$ for $x > 0$ and $y > 0$.

Use implicit differentiation to find $\dfrac{dy}{dx}$.

Hence find an equation of the tangent to the curve where $x = 1$.

Marking Instructions

$xy^2 + 3x^2y = 4$	
$y^2 + 2xy\dfrac{dy}{dx} + 6xy + 3x^2\dfrac{dy}{dx} = 0$	1M,1
$(2xy + 3x^2)\dfrac{dy}{dx} = -y^2 - 6xy$	
$\dfrac{dy}{dx} = \dfrac{-y^2 - 6xy}{2xy + 3x^2}$	1
When $x = 1$,	
$y^2 + 3y = 4 \Rightarrow y^2 + 3y - 4 = 0 \Rightarrow (y + 4)(y - 1) = 0$	
$\Rightarrow y = 1$ since $y > 0$	1
Hence at $(1,\ 1)$	
$\dfrac{dy}{dx} = \dfrac{-7}{5}$	1
Tangent is	
$(y - 1) = -\frac{7}{5}(x - 1)$	1
$5y + 7x = 12.$	
Alternative for the first 3 marks.	
$xy^2 + 3x^2y = 4$	
$y^2 + 3xy = \dfrac{4}{x}$	
$2y\dfrac{dy}{dx} + 3y + 3x\dfrac{dy}{dy} = -\dfrac{4}{x^2}$	1
$(2y + 3x)\dfrac{dy}{dx} = -\dfrac{4}{x^2} - 3y$	1
$\dfrac{dy}{dx} = \dfrac{-\frac{4}{x^2} - 3y}{2y + 3x}$	1

2007

Q2b – 3 marks

Obtain the derivative of each of the following functions:

(*a*) $f(x) = \exp(\sin 2x)$;

(*b*) $y = 4^{(x^2+1)}$.

Marking Instructions

(a)

$$f(x) = \exp(\sin 2x)$$

$$f'(x) = 2\cos 2x \exp(\sin 2x) \qquad \textbf{M1,2E1}$$

(b)

$$y = 4^{(x^2+1)}$$

$$\ln y = \ln\left(4^{(x^2+1)}\right) = (x^2+1)\ln 4 \qquad \textbf{M1}$$

$$\frac{1}{y}\frac{dy}{dx} = 2x\ln 4 \qquad \textbf{1}$$

$$\frac{dy}{dx} = 2x\ln 4 \,.\, 4^{(x^2+1)} \qquad \textbf{1}$$

Alternative:

$$y = 4^{(x^2+1)}$$

$$4 = e^{\ln 4} \qquad \textbf{1}$$

$$y = e^{\ln 4(x^2+1)}$$

$$\frac{dy}{dx} = \ln 4\, 2x\, e^{\ln 4(x^2+1)} \qquad \textbf{1,1}$$