Further Integration

2013

Q8 – 5 marks

Use integration by parts to obtain $\int x^2 \cos 3x \, dx$.

Marking Instructions

$\left[x^2.\frac{1}{3}\sin 3x\right] - \int \frac{2}{3}x \sin 3x \, dx$	•[1] Evidence of integration by parts.[1] •[2] Correct choice of u, v'. •[3] Accuracy of both expressions.
$= \left[\frac{1}{3}x^2 \sin 3x\right] - \left[-\frac{2}{9}x\cos 3x - \int -\frac{2}{9}\cos 3x \, dx\right]$	•[4] Correct second application.
$= \frac{1}{3}x^2 \sin 3x + \frac{2}{9}x \cos 3x - \frac{2}{27}\sin 3x + c$	•[5] Final integration and simplification.[4]

2012

Q11b – 4 marks

ı) Write down the derivative of $\sin^{-1}x$.

›) Use integration by parts to obtain $\int \sin^{-1}x.\frac{x}{\sqrt{1-x^2}}\,dx$.

Marking Instructions

(a) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ **1**

(b) $\int \sin^{-1}x.\frac{x}{\sqrt{1-x^2}}\,dx =$

$\sin^{-1}x\int\frac{x}{\sqrt{1-x^2}}\,dx - \int\left(\frac{d}{dx}(\sin^{-1}x)\int\frac{x}{\sqrt{1-x^2}}\,dx\right)dx$ **1**

$= \sin^{-1}x\int\frac{x}{\sqrt{1-x^2}}\,dx - \int\left(\frac{1}{\sqrt{1-x^2}}\int\frac{x}{\sqrt{1-x^2}}\,dx\right)dx$

$= \sin^{-1}x(-\sqrt{1-x^2}) - \int\left(\frac{1}{\sqrt{1-x^2}}(-\sqrt{1-x^2})\right)dx$ **1** for $\int\frac{x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2}$

$= \sin^{-1}x(-\sqrt{1-x^2}) - \int(-1)\,dx$ **1**

$= x - \sin^{-1}x.\sqrt{1-x^2} + c$ **1**

2011

Q16a – 3 marks

Define $I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,dx$ for $n \geq 1$.

(a) Use integration by parts to show that

$$I_n = \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx.$$

Marking Instructions

(a) $I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,dx$		
$= \int_0^1 1 \times (1+x^2)^{-n}\,dx$	**1**	for showing that 1 is integrated
$= \left[(1+x^2)^{-n}\int 1\,dx\right]_0^1 + \int_0^1\left(2nx(1+x^2)^{-n-1}\int 1\,dx\right)dx$	**1**	
$= \left[x(1+x^2)^{-n}\right]_0^1 + \int_0^1 2nx^2(1+x^2)^{-n-1}\,dx$		
$= \frac{1}{2^n} - 0 + 2n\int_0^1 x^2(1+x^2)^{-n-1}\,dx$	**1**	
$= \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx.$		

2010

Q3b – 4 marks

Integrate $x^2 \ln x$ with respect to x. **4**

Marking Instructions

$\int x^2 \ln x\,dx = \int (\ln x)x^2 dx$	**1M**	for using integration by parts
$= \ln x \int x^2 dx - \int \frac{1}{x}\frac{x^3}{3}dx$	**1**	for differentiating $\ln x$
$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx$		
$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c$	**1,1**	

2009

Q9 – 5 marks

Use integration by parts to obtain the exact value of $\int_0^1 x \tan^{-1} x^2\,dx$.

Marking Instructions

$\int_0^1 x\tan^{-1}x^2\,dx = \left[\tan^{-1}x^2 \int x\,dx\right]_0^1 - \int_0^1 \frac{2x}{1+x^4}\frac{x^2}{2}dx$	**1,1**
$= \left[\frac{1}{2}x^2\tan^{-1}x^2\right]_0^1 - \int_0^1 \frac{x^3}{1+x^4}dx$	
$= \left[\frac{1}{2}x^2\tan^{-1}x^2\right]_0^1 - \left[\frac{1}{4}\ln(1+x^4)\right]_0^1$	**1**
$= \frac{1}{2}\tan^{-1}1 - 0 - \left[\frac{1}{4}\ln 2 - \frac{1}{4}\ln 1\right]$	**1**
$= \frac{\pi}{8} - \frac{1}{4}\ln 2$	**1,1**

2008

Q7 – 5 marks

Use integration by parts to obtain $\int 8x^2 \sin 4x \, dx$.

Marking Instructions

$$\int 8x^2 \sin 4x \, dx = 8x^2 \int \sin 4x \, dx - \int 16x \left(\int \sin 4x \, dx \right) dx \qquad \textbf{1M,1}$$

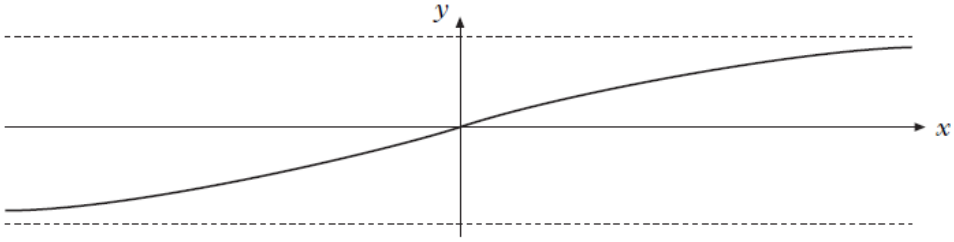
$$= 8x^2 \left(\frac{-1}{4} \cos 4x \right) - \int 16x \times \frac{-1}{4} \cos 4x \, dx \qquad \textbf{1}$$

$$= -2x^2 \cos 4x + 4\left[x \int \cos 4x \, dx - \int \frac{1}{4} \sin 4x \, dx \right] \qquad \textbf{1}$$

$$= -2x^2 \cos 4x + x \sin 4x + \tfrac{1}{4} \cos 4x + c \qquad \textbf{1}$$

2007

Q16b – 5 marks



(*a*) The diagram shows part of the graph of $f(x) = \tan^{-1} 2x$ and its asymptotes. State the equations of these asymptotes. **2**

(*b*) Use integration by parts to find the area between $f(x)$, the x-axis and the lines $x = 0$, $x = \frac{1}{2}$. **5**

(*c*) Sketch the graph of $y = |f(x)|$ and calculate the area between this graph, the x-axis and the lines $x = -\frac{1}{2}$, $x = \frac{1}{2}$. **3**

Marking Instructions

(a) $\tan^{-1} 2x$ has horizontal asymptotes at $y = \pm\frac{\pi}{2}$. **1,1**

(b)

$$\text{Area} = \int_0^{1/2} \tan^{-1} 2x \, dx \qquad \textbf{1}$$

$$= \int_0^{1/2} (\tan^{-1} 2x) \times 1 \, dx \qquad \textbf{1}$$

$$= \left[\tan^{-1} 2x \int 1.dx - \int \frac{2}{1 + 4x^2} . x \, dx\right]_0^{1/2}$$

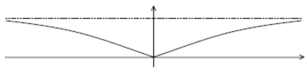
$$= \left[x \tan^{-1} 2x - \frac{1}{4} \int \frac{8x}{1 + 4x^2} dx\right]_0^{1/2}$$

$$= \left[x \tan^{-1} 2x - \frac{1}{4} \ln(1 + 4x^2)\right]_0^{1/2} \qquad \textbf{2E1}$$

$$= \left[\frac{1}{2} \tan^{-1} 1 - \frac{1}{4} \ln 2\right] - [0 - 0]$$

$$= \frac{\pi}{8} - \frac{1}{4} \ln 2 \qquad \textbf{1}$$

(c)



2E1

$$\int_{-1/2}^{1/2} |f(x)| \, dx = 2 \int_0^{1/2} \tan^{-1} 2x \, dx$$

$$= \frac{\pi}{4} - \frac{1}{2} \ln 2 \qquad \textbf{1}$$