Integration

2013

Q6 – 4 marks

Integrate $\dfrac{\sec^2 3x}{1+\tan 3x}$ with respect to x. **4**

Marking Instructions

Integrate $\dfrac{\sec^2 3x}{1+\tan 3x}$ with respect to x.	4	
$\dfrac{f'(x)}{f(x)}$		•[1] Evidence knows correct form of integral.
$=\dfrac{1}{3}\ldots$		•[2] Coefficient correct.
$\ldots\ln\ldots$		•[3] Use of ln or $\log_e$.
$\ldots\lvert 1+\tan 3x\rvert$		•[4] Completes, including use of |mod|[1]
$=\dfrac{1}{3}\ln\lvert 1+\tan 3x\rvert + c$		

OR

$u = 1 + \tan 3x$ **OR** $u = \tan 3x$		•[1] Correct substitution.
$\dfrac{du}{dx} = 3\sec^2 3x$		•[2] Differentiates accurately
$\dfrac{1}{3}du = \sec^2 3x\,dx$		
$\int \dfrac{\frac{1}{3}du}{u}$ **OR** $\int \dfrac{\frac{1}{3}du}{1+u} = \cdots$		•[3] Correct substitution of d and $f(u)$ into integral.
$=\dfrac{1}{3}\ln\lvert u\rvert + c$ **OR** $=\dfrac{1}{3}\ln\lvert 1+u\rvert + c$		
$=\dfrac{1}{3}\ln\lvert 1+\tan 3x\rvert + c$		•[4] Integrates correctly *and* substitutes back.[1,2,3]

Notes:

6.1 Do not penalise omission of "$+ c$".

6.2 |Modulus|symbols necessary for •[4]

6.3 Accept $\frac{1}{3}\log\lvert 1 + \tan 3x\rvert$ for full marks.

6.4 Accept answer without working for full marks.

6.5 Award $\ln\lvert 1 + \tan 3x\rvert$ 3 marks out of 4.

2012

Q8 – 6 marks

Use the substitution $x = 4\sin\theta$ to evaluate $\int_0^2 \sqrt{16 - x^2}\, dx$. **6**

Marking Instructions

$x = 4\sin\theta \Rightarrow dx = 4\cos\theta\, d\theta$	1	
$x = 0 \Rightarrow \theta = 0$; $x = 2 \Rightarrow \theta = \frac{\pi}{6}$	1	
$\int_0^2 \sqrt{16 - x^2}\, dx$		
$= \int_0^{\pi/6} \sqrt{16 - (4\sin\theta)^2}\,.\,4\cos\theta\, d\theta$	1	
$= \int_0^{\pi/6} \sqrt{16(1 - \sin^2\theta)}\,.\,4\cos\theta\, d\theta$		
$= \int_0^{\pi/6} \sqrt{16\cos^2\theta}\,.\,4\cos\theta\, d\theta$		
$= \int_0^{\pi/6} 16\cos^2\theta\, d\theta$	1	
$= 8\int_0^{\pi/6} (1 + \cos 2\theta)\, d\theta$		
$= 8\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/6}$	1	for applying trig. identity and integrating
$= \frac{8\pi}{6} + 4\sin\frac{\pi}{3}$		
$= \frac{4\pi}{3} + 2\sqrt{3}\ (\approx 7{\cdot}65)$	1	numerical approx. allowed

2011

Q1 – 5 marks

Express $\frac{13 - x}{x^2 + 4x - 5}$ in partial fractions and hence obtain

$$\int \frac{13 - x}{x^2 + 4x - 5}\, dx.$$

Marking Instructions

$\frac{13 - x}{x^2 + 4x - 5} = \frac{13 - x}{(x - 1)(x + 5)}$		
$= \frac{A}{x - 1} + \frac{B}{x + 5}$	1	
$13 - x = A(x + 5) + B(x - 1)$		
$x = 1 \Rightarrow 12 = 6A \Rightarrow A = 2$	1	for first value
$x = -5 \Rightarrow 18 = -6B \Rightarrow B = -3$	1	for second value
Hence $\frac{13 - x}{x^2 + 4x - 5} = \frac{2}{x - 1} - \frac{3}{x + 5}$		
$\int \frac{13 - x}{x^2 + 4x - 5}\, dx = \int \frac{2}{x - 1}\, dx - \int \frac{3}{x + 5}\, dx$		
$= 2\ln\lvert x - 1\rvert - 3\ln\lvert x + 5\rvert + c$	1	for logs
	1	for moduli

2011

Q11 – 7 marks

(*a*) Obtain the exact value of $\int_0^{\pi/4} (\sec x - x)(\sec x + x)dx$. **3**

(*b*) Find $\int \frac{x}{\sqrt{1-49x^4}}dx$. **4**

Marking Instructions

(a) $\int_0^{\pi/4}(\sec x - x)(\sec x + x)dx = \int_0^{\pi/4}(\sec^2 x - x^2)dx$	1	
$= \left[\tan x - \frac{x^3}{3}\right]_0^{\frac{\pi}{4}}$	1	
$= \left[1 - \frac{1}{3}\frac{\pi^3}{64}\right] - [0]$		
$= 1 - \frac{\pi^3}{192}$.	1	Exact value only
(b) *Method 1*		
Let $u = 7x^2$,	M1	
then $du = 14x\,dx$.	1	
$\int \frac{x}{\sqrt{1 - 49x^4}}dx = \frac{1}{14}\int \frac{du}{\sqrt{1 - u^2}}$	1	
$= \frac{1}{14}\sin^{-1} u + c$		
$= \frac{1}{14}\sin^{-1} 7x^2 + c$	1	must be in terms of x

Method 2		
$\int \frac{x}{\sqrt{1 - 49x^4}}dx = \frac{1}{14}\int \frac{14x\,dx}{\sqrt{1 - (7x^2)^2}}$	1 1 1	for fraction for numerator for $(7x^2)^2$
$= \frac{1}{14}\sin^{-1} 7x^2 + c$	1	must be in terms of x

2010

Q3a – 3 marks

(*a*) Use the substitution $t = x^4$ to obtain $\int \frac{x^3}{1+x^8}dx$. **3**

Marking Instructions

$t = x^4 \Rightarrow dt = 4x^3dx$	1	correct differential
$\int \frac{x^3}{1 + x^8}dx = \frac{1}{4}\int \frac{4x^3}{1 + (x^4)^2}dx$		
$= \frac{1}{4}\int \frac{1}{1 + t^2}dt$	1	correct integral in t
$= \frac{1}{4}\tan^{-1} t + c$		
$= \frac{1}{4}\tan^{-1} x^4 + c$	1	correct answer

2010

Q7 – 6 marks

Evaluate

$$\int_1^2 \frac{3x+5}{(x+1)(x+2)(x+3)}\,dx$$

expressing your answer in the form $\ln\frac{a}{b}$, where a and b are integers.

Marking Instructions

$\int_1^2 \frac{3x+5}{(x+1)(x+2)(x+3)}dx$ $\frac{3x+5}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$ $3x + 5 =$ $A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$	**1M**	
$x = -1 \Rightarrow 2 = 2A \Rightarrow A = 1$	**1**	for first correct coefficient
$x = -2 \Rightarrow -1 = -B \Rightarrow B = 1$	**1**	for second correct coefficient
$x = -3 \Rightarrow -4 = 2C \Rightarrow C = -2$		
Hence $\frac{3x+5}{(x+1)(x+2)(x+3)} = \frac{1}{x+1} + \frac{1}{x+2} - \frac{2}{x+3}$ $\int_1^2 \frac{3x+5}{(x+1)(x+2)(x+3)}dx = \int_1^2 (\frac{1}{x+1} + \frac{1}{x+2} - \frac{2}{x+3})dx$	**1**	for last coefficient and applying them
$= [\ln(x+1) + \ln(x+2) - 2\ln(x+3)]_1^2$ $= \ln 3 + \ln 4 - 2\ln 5 - \ln 2 - \ln 3 + 2\ln 4$	**1**	for correct integration and substitution
$= \ln\frac{3 \times 4 \times 4^2}{5^2 \times 2 \times 3} = \ln\frac{32}{25}$	**1**	

2010

Q15 – 10 marks

A new board game has been invented and the symmetrical design on the board is made from four identical "petal" shapes. One of these petals is the region enclosed between the curves $y = x^2$ and $y^2 = 8x$ as shown shaded in diagram 1 below.

Calculate the area of the complete design, as shown in diagram 2. **5**

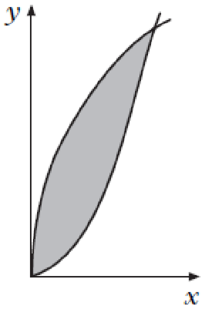

Diagram 1

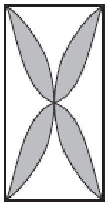
Diagram 2

The counter used in the game is formed by rotating the shaded area shown in diagram 1 above, through 360° about the y-axis. Find the volume of plastic required to make one counter. **5**

Marking Instructions

$(x^2)^2 = 8x \Rightarrow x^4 = 8x \Rightarrow x = 0,\ 2$	**1**	values of x
Area $= 4\int_0^2 (\sqrt{8x} - x^2)\,dx$	**1M**	$4\int_0^2$
	1	the rest
$= 4\left[\sqrt{8}\left(\frac{2}{3}x^{3/2}\right) - \frac{1}{3}x^3\right]_0^2$	**1**	
$= 4\left[\frac{16}{3} - \frac{8}{3}\right] = \frac{32}{3}$	**1**	
Volume of revolution about the y-axis $= \pi\int x^2 dy$.	**1M**	
So in this case, we need to calculate two volumes and subtract:		
$V = \pi\left[\int_0^4 y\,dy\right] - \pi\left[\int_0^4 \frac{y^4}{64}dy\right]$	**1,1**	each term
$= \pi\left[\frac{y^2}{2}\right]_0^4 - \pi\left[\frac{y^5}{320}\right]_0^4$	**1**	
$= \pi\left[8 - \frac{64 \times 4^2}{320}\right]$		
$= \frac{40 - 16}{5}\pi\left(= \frac{24\pi}{5}\right)(\approx\ 15)$	**1**	

2009

Q5 – 4 marks

Show that

$$\int_{\ln\frac{3}{2}}^{\ln 2} \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \ln\frac{9}{5}. \qquad \mathbf{4}$$

Marking Instructions

Method 1

$$\int_{\ln\frac{3}{2}}^{\ln 2} \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$$

Let $u = e^x - e^{-x}$, then $du = (e^x + e^{-x})\,dx$. **1**

When $x = \ln\frac{3}{2}$, $u = \frac{3}{2} - \frac{2}{3} = \frac{5}{6}$ and when $x = \ln 2$, $u = 2 - \frac{1}{2} = \frac{3}{2}$. **1**

$$\int_{\ln\frac{3}{2}}^{\ln 2} \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \int_{5/6}^{3/2} \frac{du}{u}$$

$$= [\ln u]_{5/6}^{3/2} \qquad \mathbf{1}$$

$$= \ln\frac{3}{2} - \ln\frac{5}{6} = \ln\frac{9}{5} \qquad \mathbf{1}$$

Method 2

$$\int_{\ln\frac{3}{2}}^{\ln 2} \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = [\ln(e^x - e^{-x})]_{\ln\frac{3}{2}}^{\ln 2} \qquad \mathbf{1,1}$$

$$= \ln\left(2 - \frac{1}{2}\right) - \ln\left(\frac{3}{2} - \frac{2}{3}\right) \qquad \mathbf{1}$$

$$= \ln\frac{3}{2} - \ln\frac{5}{6} = \ln\frac{9}{5} \qquad \mathbf{1}$$

2009

Q7 – 6 marks

Use the substitution $x = 2\sin\theta$ to obtain the exact value of $\displaystyle\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4 - x^2}}\,dx$. **6**

(Note that $\cos 2A = 1 - 2\sin^2 A$.)

Marking Instructions

$$x = 2\sin\theta \Rightarrow dx = 2\cos\theta\,d\theta \qquad \mathbf{1}$$

$$x = 0 \Rightarrow \theta = 0;\; x = \sqrt{2} \Rightarrow \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \qquad \mathbf{1}$$

$$\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4 - x^2}}\,dx = \int_0^{\pi/4} \frac{4\sin^2\theta}{\sqrt{4 - 4\sin^2\theta}}(2\cos\theta)\,d\theta$$

$$= \int_0^{\pi/4} \frac{4\sin^2\theta}{2\cos\theta}(2\cos\theta)\,d\theta \qquad \mathbf{1}$$

$$= 2\int_0^{\pi/4} (2\sin^2\theta)\,d\theta$$

$$= 2\int_0^{\pi/4} (1 - \cos 2\theta)\,d\theta \qquad \mathbf{1}$$

$$= 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} \qquad \mathbf{1}$$

$$= 2\left\{\left[\frac{\pi}{4} - \frac{1}{2}\right] - 0\right\}$$

$$= \frac{\pi}{2} - 1 \qquad \mathbf{1}$$

2008

Q2 – 5 marks

(*a*) Differentiate $f(x) = \cos^{-1}(3x)$ where $-\frac{1}{3} < x < \frac{1}{3}$. **2**

(*b*) Given $x = 2\sec\theta$, $y = 3\sin\theta$, use parametric differentiation to find $\frac{dy}{dx}$ in terms of θ. **3**

Marking Instructions

(a)

$$f(x) = \cos^{-1}(3x)$$

$$f'(x) = \frac{-1}{\sqrt{1-(3x)^2}} \times 3 \qquad \textbf{1,1}$$

$$= \frac{-3}{\sqrt{1-9x^2}}$$

(b)

$$\frac{dx}{d\theta} = 2\sec\theta\tan\theta, \qquad \frac{dy}{d\theta} = 3\cos\theta \qquad \textbf{1,1}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos\theta}{2\sec\theta\tan\theta} \qquad \textbf{1}$$

$$= \frac{3\cos^3\theta}{2\sin\theta}$$

2007

Q10 – 6 marks

Use the substitution $u = 1 + x^2$ to obtain $\int_0^1 \frac{x^3}{(1+x^2)^4}\,dx$. **5**

A solid is formed by rotating the curve $y = \frac{x^{3/2}}{(1+x^2)^2}$ between $x = 0$ and $x = 1$ through $360°$ about the x-axis. Write down the volume of this solid. **1**

Marking Instructions

$$1 + x^2 = u \Rightarrow 2x\,dx = du \qquad \mathbf{1}$$

$$x = 0 \Rightarrow u = 1; \qquad x = 1 \Rightarrow u = 2 \qquad \mathbf{1}$$

$$\int_0^1 \frac{x^3}{(1+x^2)^4}\,dx = \int_1^2 \frac{(u-1)}{2u^4}\,du \qquad \mathbf{1}$$

$$= \frac{1}{2}\int_1^2 (u^{-3} - u^{-4})\,du$$

$$= \frac{1}{2}\left[-\frac{1}{2}u^{-2} + \frac{1}{3}u^{-3}\right]_1^2 \qquad \mathbf{1}$$

$$= \frac{1}{2}\left[-\frac{1}{8} + \frac{1}{24}\right] - \frac{1}{2}\left[-\frac{1}{2} + \frac{1}{3}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{12} + \frac{1}{6}\right] = \frac{1}{24} \qquad \mathbf{1}$$

The volume of revolution is given by $V = \int_a^b \pi y^2 dx$. So in this case

$$V = \pi \int_0^1 \frac{x^3}{(1+x)^4}\,dx = \frac{\pi}{24}. \qquad \mathbf{1}$$

Integration by parts could be used for marks three, four and five.

$$\int_1^2 \frac{u-1}{2u^4}\,du = \frac{1}{2}\left[(u-1)\int u^{-4}du - \int 1.\frac{u^{-3}}{-3}du\right]_1^2 \qquad \mathbf{1}$$

$$= \frac{1}{2}\left[\frac{u-1}{-3u^3} + \frac{u^{-2}}{(-6)}\right]_1^2 \qquad \mathbf{1}$$

$$= \frac{1}{2}\left[\frac{1}{-24} - \frac{1}{24}\right] - \frac{1}{2}\left[0 - \frac{1}{6}\right]$$

$$= -\frac{1}{24} + \frac{1}{12} = \frac{1}{24} \qquad \mathbf{1}$$