Maclaurin Theorem

2012

Q6 – 5 marks

Write down the Maclaurin expansion of e^x as far as the term in x^3. **1**

Hence, or otherwise, obtain the Maclaurin expansion of $(1 + e^x)^2$ as far as the term in x^3. **4**

Written Solutions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

If you can remember this you can remember sin & cos?

$$(1 + e^x)^2 = \left(2 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right)^2$$

$$= \left(2 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(2 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)$$

$$\begin{aligned} = \; & 4 + 2x + x^2 + \frac{x^3}{3} \\ & \quad\;\; 2x + x^2 + \frac{1}{2}x^3 \\ & \qquad\quad\;\; x^2 + \frac{x^3}{2} \\ & \qquad\qquad\quad\;\; \frac{x^3}{3} \end{aligned}$$

$$= 4 + 4x + 3x^2 + \frac{2 + 3 + 3 + 2}{6}x^3$$

$$= 4 + 4x + 3x^2 + \frac{5}{3}x^3 + \dots$$

2011

Q5 – 6 marks

Obtain the first four terms in the Maclaurin series of $\sqrt{1+x}$, and hence write down the first four terms in the Maclaurin series of $\sqrt{1+x^2}$. **4**

Hence obtain the first four terms in the Maclaurin series of $\sqrt{(1+x)(1+x^2)}$. **2**

Written Solutions

$\sqrt{1+x}$

Let $f(x) = (1+x)^{\frac{1}{2}}$ $\qquad f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$ $\qquad f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$ $\qquad f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}(1+x)^{-5/2}$ $\qquad f'''(0) = \frac{3}{8}$

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \frac{f^{n}(0)\,x^n}{n!} = \frac{1x^0}{0!} + \frac{\frac{1}{2}x^2}{1!} + \frac{\left(-\frac{1}{4}\right)x^2}{2!} + \frac{\left(\frac{3}{8}\right)x^3}{3!} + \dots$$

$$\sqrt{1+x} = \underline{\underline{1 + \frac{1}{2}x^2 - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots}}$$

$$\sqrt{1+x^2} = \underline{\underline{1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6}}$$

$\sqrt{(1+x)(1+x^2)}$

$= \sqrt{1+x} \cdot \sqrt{1+x^2}$

$= \left(1 + \frac{1}{2}x^1 - \frac{1}{8}x^2 + \frac{1}{16}x^3\right)\left(1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6\right)$

take as far as x^3

$= 1 + \frac{1}{2}x^2 \qquad \times 1$

$\quad \frac{1}{2}x + \frac{1}{4}x^3 \qquad \times \frac{1}{2}x$

$\quad - \frac{1}{8}x^2 \qquad \times -\frac{1}{8}x^2$

$\quad + \frac{1}{16}x^3 \qquad \times \frac{1}{16}x^3$

$$= \underline{\underline{1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots}}$$

2010

Q9 – 4 marks

Obtain the first three non-zero terms in the Maclaurin expansion of $(1 + \sin^2 x)$. **4**

Written Solutions

$$(1 + \sin^2 x)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} -$$

So,
$$1 + \sin^2 x$$

$$= 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)^2$$

$$= 1 + \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)$$

$$= 1 + x^2 - \frac{x^4}{6} - \frac{x^4}{6}$$

$$= \underline{\underline{1 + x^2 - \frac{x^4}{3}}}$$

[PTO for first principles sol^n]

$f(x) = 1 + \sin^2 x$

$f'(x) = 2\sin x \cos x$

$= \sin 2x$

$f''(x) = 2\cos 2x$

$f'''(x) = -4\sin 2x$

$f''''(x) = -8\cos 2x$

$f(0) = 1 + 0 = \underline{\underline{1}}$

$f'(0) = 0$

$f''(0) = 2 \times 1 = \underline{\underline{2}}$

$f'''(0) = 0$

$f''''(0) = -8 \times 1 = -\underline{\underline{8}}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)} x^n}{n!}$$

$$= \frac{1x^0}{0!} + \frac{2x^2}{2!} - \frac{8x^4}{4!}$$

$$= \underline{\underline{1 + x^2}} - \frac{x^4}{3}$$

2009

Q14 – 9 marks

Express $\dfrac{x^2+6x-4}{(x+2)^2(x-4)}$ in partial fractions. **4**

Hence, or otherwise, obtain the first three non-zero terms in the Maclaurin expansion of $\dfrac{x^2+6x-4}{(x+2)^2(x-4)}$. **5**

Written Solutions

$$\frac{x^2+6x-4}{(x+2)^2(x-4)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-4}$$

$$x^2+6x-4 = A(x+2)(x-4) + B(x-4) + C(x+2)^2$$

Let $x=4$ $\quad 36 = 36C$

$\underline{\underline{C=1}}$

Let $x=-2$ $\quad -12 = -6B$

$\underline{\underline{B=2}}$

Let $x=0$ (with $C=1$ $B=2$) $\quad -4 = A(2)(-4) + 2(-4) + (1)(4)$

$-4 = -8A - 4$

$\underline{\underline{A=0}}$

So, $$\frac{x^2+6x-4}{(x-2)^2(x-4)} = \underline{\underline{\frac{2}{(x+2)^2} + \frac{1}{x-4}}}$$

PTO

$$f(x) = \frac{x^2 + 6x \quad 4}{(x+2)^2(x-4)} = \frac{2}{(x+2)^2} + \frac{1}{x-4}$$

$$f(x) = 2(x+2)^{-2} + (x-4)^{-1} \qquad f(0) = \frac{1}{4}$$

$$f'(x) = -4(x+2)^{-3} - (x-4)^{-2} \qquad f'(0) = -\frac{9}{16}$$

$$f''(x) = 12(x+2)^{-4} + 2(x-4)^{-3} \qquad f''(0) = \frac{23}{32}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$$

$$= \frac{\frac{1}{4}x^0}{0!} + \frac{\left(-\frac{9}{16}\right)x^1}{1!} + \frac{\left(\frac{23}{32}\right)x^2}{2!}$$

$$= \frac{1}{4} - \frac{9}{16}x + \frac{23}{64}x^2 + \dots$$

2008

Q12 – 7 marks

Obtain the first three non-zero terms in the Maclaurin expansion of $x \ln(2 + x)$. **3**

Hence, or otherwise, deduce the first three non-zero terms in the Maclaurin expansion of $x \ln(2 - x)$. **2**

Hence obtain the first **two** non-zero terms in the Maclaurin expansion of $x \ln(4 - x^2)$. **2**

[*Throughout this question, it can be assumed that* $-2 < x < 2$.]

Written Solutions

$$f(x) = x\ln(2+x) \qquad f(0) = 0$$

$u = x$, $u' = 1$, $v = \ln(2+x)$, $v' = \frac{1}{2+x}$

$$f'(x) = \ln(2+x) + \frac{x}{2+x} \qquad f'(0) = \ln 2$$

$u = x$, $u' = 1$, $v = 2+x$, $v' = 1$

$$f''(x) = \frac{1}{2+x} + \frac{2+x-x}{(2+x)^2} \qquad f''(0) = 1$$

$u = 2$, $u' = 0$, $v = (2+x)^2$, $v' = 2(2+x) = 4+2x$

$$f'''(x) = -(2+x)^{-2} + \frac{0-8-4x}{(2+x)^4} \qquad f'''(0) = -\frac{3}{4}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(0)\,x^n}{n!}$$

$$= \frac{(\ln 2)x^1}{1!} + \frac{(1)x^2}{2!} + \frac{(-\frac{3}{4})x^3}{3!}$$

$$= \ln(2)x + \frac{x^2}{2} - \frac{x^3}{8}$$

$$f(x) = x\ln(2-x) = -(-x)\ln(2+(-x))$$

$$= -\left[(\ln 2)(-x) + \frac{1}{2}(-x)^2 - \frac{1}{8}(-x)^3 + \dots\right]$$

$$= (\ln 2)x - \frac{1}{2}x^2 - \frac{1}{8}x^3 + \dots$$

$$x\ln(4-x^2) = x\ln(2-x)(2+x)$$

$$= x\ln(2+x) + x\ln(2-x)$$

$$= (\ln 2)x + \frac{1}{2}x^2 - \frac{1}{8}x^3 + (\ln 2)x - \frac{1}{2}x^2 - \frac{1}{8}x^3$$

$$= 2(\ln 2)x - \frac{1}{4}x^3 + \dots$$

$$= (\ln 4)x - \frac{1}{4}x^3$$

2007

Q6 – 5 marks

Find the Maclaurin series for $\cos x$ as far as the term in x^4. 2

Deduce the Maclaurin series for $f(x) = \frac{1}{2}\cos 2x$ as far as the term in x^4. 2

Hence write down the first three non-zero terms of the series for $f(3x)$. 1

cont. "recall"

Written Solutions

$f(x) = \cos x$ $\quad f(0) = 1$ ✓

$f'(x) = -\sin x$ $\quad f'(0) = 0$

$f''(x) = -\cos x$ $\quad f''(0) = -1$ ✓

$f'''(x) = \sin x$ $\quad f'''(0) = 0$

$f^{iv}(x) = \cos x$ $\quad f^{iv}(0) = 1$ ✓

So, $\cos x = \sum_{n=0}^{\infty} \frac{f^n(0)x^n}{n!}$

$= \frac{1x^0}{0!} + \frac{(-1)x^2}{2!} + \frac{1x^4}{4!}$

$= 1 - \frac{x^2}{2} + \frac{x^4}{24}$

$f(x) = \frac{1}{2}\cos 2x$

$= \frac{1}{2}\left(1 - \frac{1}{2}(2x)^2 + \frac{1}{24}(2x)^4\right)$

$= \frac{1}{2}\left(1 - 2x^2 + \frac{2}{3}x^4\right)$

$= \frac{1}{2} - x^2 + \frac{1}{3}x^4$

$f(3x) = \frac{1}{2} - (3x)^2 + \frac{1}{3}(3x)^4$

$= \frac{1}{2} - 9x^2 + 27x^4$