Proofs

2013

Q9 – 6 marks

Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n}\left(4r^3+3r^2+r\right)=n\left(n+1\right)^3 \qquad . \qquad\qquad 6$$

Marking Instructions

Prove by induction that, for all positive integers n, $\sum_{r=1}^{n}\left(4r^3+3r^2+r\right)=n\left(n+1\right)^3$.	**6**	
For $n=1$ L.H.S $\sum_{r=1}^{n}\left(4r^3+3r^2+r\right)=4+3+1=8$ R.H.S $n\left(n+1\right)^3=1\times 2^3=8$ $\Rightarrow$ true for $n=1$		•[1] Evaluation of both sides independently to 8.[8]
Assume true for $n=k$, $\sum_{r=1}^{k}\left(4r^3+3r^2+r\right)=k\left(k+1\right)^3$ Consider $n=k+1$, $\sum_{r=1}^{k+1}\left(4r^3+3r^2+r\right)$		•[2] Inductive hypothesis (must include "Assume true…" or equivalent phrase).[3,4]
$=\sum_{r=1}^{k}\left(4r^3+3r^2+r\right)+4\left(k+1\right)^3+3\left(k+1\right)^2+\left(k+1\right)$		•[3] Addition of $(k+1)$th term.[2]
$=k\left(k+1\right)^3+4\left(k+1\right)^3+3\left(k+1\right)^2+\left(k+1\right)$ $=\left(k+1\right)\left[k\left(k+1\right)^2+4\left(k+1\right)^2+3\left(k+1\right)+1\right]$		•[4] Use of inductive hypothesis *and* first step in factorisation process.[1,6]
$=\left(k+1\right)\left[k\left(k^2+2k+1\right)+4\left(k^2+2k+1\right)+3\left(k+1\right)+1\right]$ $=\left(k+1\right)\left[k^3+2k^2+k+4k^2+8k+4+3k+3+1\right]$ $=\left(k+1\right)\left(k^3+6k^2+12k+8\right)$		•[5] Processing and simplifying to arrive at second factor.[1]
$=\left(k+1\right)\left(k+2\right)^3$ $=\left(k+1\right)\left(\left(k+1\right)+1\right)^3$ Hence, if true for $n=k$, then true for $n=k+1$, but since true for $n=1$, then by induction true for all positive integers n.		•[6] Statement of result in terms of $(k+1)$ and valid statement of conclusion.[1,7]

2013

Q12 – 4 marks

Let n be a natural number.
For each of the following statements, decide whether it is true or false.
If true, give a proof; if false, give a counterexample.

A If n is a multiple of 9 then so is n^2.

B If n^2 is a multiple of 9 then so is n. **4**

Marking Instructions

	Let n be a natural number. For each of the following statements, decide whether it is true or false. If true, give a proof, if false, give a counterexample. **A** **If n is a multiple of 9 then so is n^2.** **B** **If n^2 is a multiple of 9 then so is n.**	**4**		
A	Suppose $n = 9m$ for some natural number [positive integer], m.		•[1]	Generalisation, using *different* letter.[3, 6]
	Then $n^2 = 81m^2 = 9(9m^2)$		•[2]	Correct multiplication *and* 9 extracted as a factor.
	Hence n^2 is a multiple of 9, so **A** is **true**.		•[3]	Conclusion of proof *and* state A true.[1]
B	**False.** Accept any valid counterexample: $n = 3, 6, 12, 15, 21$ etc		•[4]	Valid counterexample *and* conclusion.[5]

2012

Q16a – 6 marks

(a) Prove by induction that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all integers $n \geq 1$. **6**

Marking Instructions

(a) For $n = 1$, the LHS $= \cos\theta + i\sin\theta$ and the RHS $= \cos\theta + i\sin\theta$. Hence the result is true for $n = 1$.	**1**	
Assume the result is true for $n = k$, i.e. $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$.	**1**	working with n is penalised.
Now consider the case when $n = k + 1$: $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$	**1**	
$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$	**1**	for applying the inductive hypothesis
$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$	**1**	multiplying and collecting
$= \cos(k+1)\theta + i\sin(k+1)\theta$ Thus, if the result is true for $n = k$ the result is true for $n = k + 1$. Since it is true for $n = 1$, the result is true for all $n \geqslant 1$.	**1**	

2011

Q12 – 5 marks

Prove by induction that $8^n + 3^{n-2}$ is divisible by 5 **for all integers** $\boldsymbol{n} \geq 2$. **5**

Marking Instructions

For $n = 2$, $8^2 + 3^0 = 64 + 1 = 65$. True when $n = 2$.	1	
Assume true for k, i.e. that $8^k + 3^{k-2}$ is divisible by 5, i.e. can be expressed as $5p$ for an integer p.	1	for the inductive hypothesis
Now consider $8^{k+1} + 3^{k-1}$		
$= 8 \times 8^k + 3^{k-1}$	1	
$= 8 \times (5p - 3^{k-2}) + 3^{k-1}$	1	for replacing 8^k
$= 40p - 3^{k-2}(8 - 3)$		
$= 5(8p - 3^{k-2})$ which is divisible by 5.	1	
So, since it is true for $n = 2$, it is true for all $n \geqslant 2$.		

2011

Q16b – 5 marks

Define $I_n = \int_0^1 \frac{1}{(1+x^2)^n} dx$ for $n \geq 1$.

(*a*) Use integration by parts to show that

$$I_n = \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}} dx. \qquad 3$$

(*b*) Find the values of A and B for which

$$\frac{A}{(1+x^2)^n} + \frac{B}{(1+x^2)^{n+1}} = \frac{x^2}{(1+x^2)^{n+1}}$$

and hence show that

$$I_{n+1} = \frac{1}{n \times 2^{n+1}} + \left(\frac{2n-1}{2n}\right) I_n. \qquad 5$$

(*c*) Hence obtain the exact value of $\int_0^1 \frac{1}{(1+x^2)^3} dx$. 3

Marking Instructions

(a) $I_n = \int_0^1 \frac{1}{(1+x^2)^n} dx$		
$= \int_0^1 1 \times (1+x^2)^{-n} dx$	1	for showing that 1 is integrated
$= \left[(1+x^2)^{-n} \int 1\, dx\right]_0^1 + \int_0^1 \left(2nx(1+x^2)^{-n-1} \int 1\, dx\right) dx$	1	
$= \left[x(1+x^2)^{-n}\right]_0^1 + \int_0^1 2nx^2(1+x^2)^{-n-1} dx$		
$= \frac{1}{2^n} - 0 + 2n\int_0^1 x^2(1+x^2)^{-n-1} dx$	1	
$= \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}} dx.$		

(b) $\frac{A}{(1+x^2)^n} + \frac{B}{(1+x^2)^{n+1}} = \frac{x^2}{(1+x^2)^{n+1}}$		
$\Rightarrow A(1+x^2) + B = x^2$	1	
$\Rightarrow A = 1, B = -1$	1	
$\frac{1}{(1+x^2)^n} + \frac{-1}{(1+x^2)^{n+1}} = \frac{x^2}{(1+x^2)^{n+1}}$ (*)		
$I_n = \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}} dx.$		
$= \frac{1}{2^n} + 2n\int_0^1 \frac{1}{(1+x^2)^n} dx + 2n\int_0^1 \frac{-1}{(1+x^2)^{n+1}} dx$	1	Using (*)
$= \frac{1}{2^n} + 2nI_n - 2nI_{n+1}$	1	Recognising I_n and I_{n+1}
$2nI_{n+1} = \frac{1}{2^n} + (2n-1)I_n$	1	
$I_{n+1} = \frac{1}{n \times 2^{n+1}} + \left(\frac{2n-1}{2n}\right)I_n.$		

(c) $\int_0^1 \frac{1}{(1+x^2)^3} dx = I_3$		
$= \frac{1}{16} + \frac{3}{4}I_2$	1	
$= \frac{1}{16} + \frac{3}{4}\left(\frac{1}{4} + \frac{1}{2}I_1\right)$		
$= \frac{1}{4} + \frac{3}{8}\int_0^1 \frac{1}{1+x^2} dx$		
$= \frac{1}{4} + \frac{3}{8}\left[\tan^{-1} x\right]_0^1$	1	
$= \frac{1}{4} + \frac{3}{8}\frac{\pi}{4} = \frac{1}{4} + \frac{3\pi}{32}.$	1	

2010

Q8 – 6 marks

(*a*) Prove that the product of two odd integers is odd. **2**

(*b*) Let p be an odd integer. Use the result of (*a*) to prove by induction that p^n is odd for all positive integers n. **4**

Marking Instructions

(a) Write the odd integers as: $2n + 1$ and $2m + 1$ where n and m are integers.	**1M**	for unconnected odd integers
Then $(2n+1)(2m+1) = 4nm + 2n + 2m + 1$		
$= 2(2nm + n + m) + 1$	**1**	demonstrating clearly
which is odd.		
(b) Let $n = 1$, $p^1 = p$ which is given as odd.	**1**	
Assume p^k is odd and consider p^{k+1}.	**1M**	
$p^{k+1} = p^k \times p$	**1**	
Since p^k is assumed to be odd and p is odd, p^{k+1} is the product of two odd integers is therefore odd.	**1**	for a valid explanation from a previous correct argument
Thus p^{n+1} is an odd integer for all n if p is an odd integer.		

2010

Q12 – 4 marks

Prove by contradiction that if x is an irrational number, then $2 + x$ is irrational. **4**

Marking Instructions

Assume $2 + x$ is rational	**1**	
and let $2 + x = \frac{p}{q}$ where p, q are integers.	**1**	
So $x = \frac{p}{q} - 2$		
$= \frac{p - 2q}{q}$	**1**	as a single fraction
Since $p - 2q$ and q are integers, it follows that x is rational. This is a contradiction.	**1**	

2009

Q4 – 5 marks

Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1}.$$ **5**

Marking Instructions

When $n = 1$, LHS $= \frac{1}{1 \times 2} = \frac{1}{2}$, RHS $= 1 - \frac{1}{2} = \frac{1}{2}$. So true when $n = 1$. **1**

Assume true for $n = k$, $\sum_{r=1}^{k} \frac{1}{r(r+1)} = 1 - \frac{1}{k+1}$. **1**

Consider $n = k + 1$

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)}$$ **1**

$$= 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= 1 - \frac{k+2-1}{(k+1)(k+2)} = 1 - \frac{k+1}{(k+1)((k+1)+1)}$$ **1**

$$= 1 - \frac{1}{((k+1)+1)}$$ **1**

Thus, if true for $n = k$, statement is true for $n = k + 1$, and, since true for $n = 1$, true for all $n \geqslant 1$.

2008

Q11 – 5 marks

For each of the following statements, decide whether it is true or false and prove your conclusion.

A For all natural numbers m, if m^2 is divisible by 4 then m is divisible by 4.

B The cube of any odd integer p plus the square of any even integer q is always odd. **5**

Marking Instructions

(a)	Counter example $m = 2$.	**1,1**
	So statement is false.	
(b)	Let the numbers be $2n + 1$ and $2m$ then	**1M**
	$(2n+1)^3 + (2m)^2 = 8n^3 + 12n^2 + 6n + 1 + 4m^2$	**1**
	$= 2(4n^3 + 6n^2 + 3n + 2m^2) + 1$	**1**
	which is odd.	
	OR	
	Proving algebraically that either the cube of an odd number is odd or the square of an even number is even.	**1**
	Odd cubed is odd and even squared is even.	**1**
	So the sum of them is odd.	**1**

2007

Q12 – 5 marks

Prove by induction that for $a > 0$,

$$(1 + a)^n \geq 1 + na$$

for all positive integers n. **5**

Marking Instructions

Consider $n = 1$, LHS $= (1 + a)$, RHS $= 1 + a$ so true for $n = 1$.	**1**
Assume that $(1 + a)^k \geqslant 1 + ka$ and consider $(1 + a)^{k+1}$.	**1**
$(1 + a)^{k+1} = (1 + a)(1 + a)^k$	**1**
$\geqslant (1 + a)(1 + ka)$	**1**
$= 1 + a + ka + ka^2$	
$= 1 + (k + 1)a + ka^2$	
$> 1 + (k + 1)a$ since $ka^2 > 0$	**1**
as required. So since true for $n = 1$, by mathematical induction statement is true for all $n \geqslant 1$.	