Sequences and Series

2013

Q17 – 10 marks

Write down the sums to infinity of the geometric series

$1 + x + x^2 + x^3 + \ldots\ldots.$

and

$1 - x + x^2 - x^3 + \ldots\ldots.$

valid for $|x| < 1$.

Assuming that it is permitted to integrate an infinite series term by term, show that, for $|x| < 1$,

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\right). \qquad 7$$

Show how this series can be used to evaluate ln 2.

Hence determine the value of ln 2 correct to 3 decimal places. 3

Written Solutions

$1 + x + x^2 + x^3 + \ldots$

$a = 1 \quad r = x$

$S_\infty = \dfrac{a}{1-r}$

$= \dfrac{1}{1-x}$

$1 - x + x^2 - x^3 + \ldots$

$a = 1 \quad r = -x$

$S_\infty = \dfrac{1}{1-(-x)}$

$= \dfrac{1}{1+x}$

$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

$\frac{d}{dx}\left(\ln\left(\frac{1+x}{1-x}\right)\right) = \dfrac{1}{1+x} + \dfrac{1}{1-x}$

$= 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$
$+ 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots$

$= 2 + 2x^2 + 2x^4 + 2x^6 + \ldots$

$\int \frac{d}{dx}\left(\ln\left(\frac{1+x}{1-x}\right)\right)dx = 2\int(1 + x^2 + x^4 + x^6 + \ldots)\,dx$

$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right)$ as req^d

$\ln 2 \Rightarrow \quad \frac{1+x}{1-x} = 2$

$1 + x = 2 - 2x$

$x = \frac{1}{3}$

So, $\ln 2 \approx 2\left(\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 + \frac{1}{7}\left(\frac{1}{3}\right)^7\right.$

$\approx 0.69313\ldots$

$\approx \underline{0.693}$ (3 d.p.)

2012

Q2 – 5 marks

The first and fourth terms of a geometric series are 2048 and 256 respectively. Calculate the value of the common ratio. **2**

Given that the sum of the first n terms is 4088, find the value of n. **3**

Written Solutions

$u_1 = 2048 \qquad u_4 = 256$

$u_1 = a = 2048 \qquad u_4 = ar^3 = 256$

$$\frac{u_4}{u_1} = \frac{ar^3}{a} = r^3 = \frac{256}{2048}$$

$$= \frac{1}{8}$$

$$\Rightarrow \underline{\underline{r = \frac{1}{2}}}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{2048\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} = 4088$$

$$\Rightarrow \quad 2048\left(1-\frac{1}{2^n}\right) = 2044$$

$$1-\frac{1}{2^n} = \frac{2044}{2048}$$

$$\frac{1}{2^n} = 1-\frac{2044}{2048}$$

$$\frac{1}{2^n} = \frac{4}{2048}$$

$$\frac{1}{2^n} = \frac{1}{512}$$

$$\Rightarrow \quad 2^n = 512$$

$$\underline{\underline{n = 9}}$$

2011

Q8 – 4 marks

Write down an expression for $\sum_{r=1}^{n} r^3 - \left(\sum_{r=1}^{n} r\right)^2$ **1**

and an expression for

$$\sum_{r=1}^{n} r^3 + \left(\sum_{r=1}^{n} r\right)^2.$$ **3**

Written Solutions

$$\sum_{r=1}^{n} r^3 - \left(\sum_{r=1}^{n} r\right)^2$$

$$= \frac{1}{4}n^2(n+1)^2 - \left(\frac{1}{2}n(n+1)\right)^2$$

$$= \frac{1}{4}n^2(n+1)^2 - \frac{1}{4}n^2(n+1)^2$$

$$= 0$$

$$\sum_{r=1}^{n} r^3 + \left(\sum_{r=1}^{n} r\right)^2$$

$$= \frac{1}{4}n^2(n+1)^2 + \frac{1}{4}n^2(n+1)^2$$

$$= \frac{1}{2}n^2(n+1)^2$$

2011

Q13 – 9 marks

The first three terms of an arithmetic sequence are $a, \frac{1}{a}, 1$ where $a < 0$.

Obtain the value of a and the common difference. **5**

Obtain the smallest value of n for which the sum of the first n terms is greater than 1000. **4**

Written Solutions

$u_1 = a \quad u_2 = \frac{1}{a} \quad u_3 = 1$

$d = u_2 - u_1 = u_3 - u_2$ $\quad\quad$ $\therefore a, a+d, a+2d, \ldots$

$\Rightarrow \frac{1}{a} - a = 1 - \frac{1}{a}$

$\Rightarrow 1 - a^2 = a - 1$

$\Rightarrow a^2 + a - 2 = 0$

$\Rightarrow (a-1)(a+2) = 0$

$\Rightarrow a = 1$ or $a = -2$

$\Rightarrow \underline{\underline{a = -2}} \quad (a < 0) \quad \rightarrow \quad d = \frac{1}{-2} - (-2)$

$\quad\quad = \underline{\underline{\frac{3}{2}}}$

$S_n = \frac{1}{2}n\left(2a + (n-1)d\right)$

$S_n > 1000$

$\Rightarrow \frac{1}{2}n\left(2(-2) + (n-1)\frac{3}{2}\right) > 1000$

$-2n + \frac{3}{4}n^2 - \frac{3}{4}n > 1000$

$-8n + 3n^2 - 3n > 4600$

$3n^2 - 11n - 4000 > 0$

$\therefore \quad n > \frac{11 \pm \sqrt{121 + 48000}}{6}$

$n > 38.394$

<u>smallest value of n is 39.</u>

2010

Q2 – 5 marks

The second and third terms of a geometric series are –6 and 3 respectively.

Explain why the series has a sum to infinity, and obtain this sum. 5

Written Solutions

$u_2 = -6 \qquad u_3 = 3 \qquad\qquad u_n = ar^{n-1}$

$ar = -6 \qquad ar^3 = 3 \qquad\qquad a, ar, ar^2, \ldots$

$$\frac{ar^3}{ar} = r = \frac{3}{-6} = \underline{\underline{-\frac{1}{2}}}$$

$$\Rightarrow \quad a\left(-\tfrac{1}{2}\right) = -6$$

$$\underline{\underline{a = 12}}$$

Since $\left|-\frac{1}{2}\right| < 1$ $\Rightarrow$ terms will converge and have a sum to infinity

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{12}{1-\left(-\frac{1}{2}\right)}$$

$$= \underline{\underline{8}}$$

2009

Q12 – 6 marks

The first two terms of a geometric sequence are $a_1 = p$ and $a_2 = p^2$. Obtain expressions for S_n and S_{2n} in terms of p, where $S_k = \sum_{j=1}^{k} a_j$. **1,1**

Given that $S_{2n} = 65S_n$ show that $p^n = 64$. **2**

Given also that $a_3 = 2p$ and that $p > 0$, obtain the exact value of p and hence the value of n. **1,1**

Written Solutions

$a_1 = p \quad a_2 = p^2 \Rightarrow r = \frac{p^2}{p} = p \qquad a_n = p \cdot p^{n-1} = p^n$

$S_n = \frac{a(1-r^n)}{1-r} = \frac{p(1-p^n)}{1-p} \qquad S_{2n} = \frac{p(1-p^{2n})}{1-p}$

$S_{2n} = 65 S_n \Rightarrow \frac{p(1-p^{2n})}{1-p} = \frac{65p(1-p^n)}{1-p}$

$\Rightarrow 1 - p^{2n} = 65(1-p^n)$

$\Rightarrow p^{2n} - 65p^n + 64 = 0$

$\Rightarrow (p^n - 1)(p^n - 64) = 0$

$\Rightarrow p^n = 1$ or $p^n = 64 \qquad p \neq 1$

$a_3 = 2p$

$p^3 = 2p$

$p = \sqrt{2}$

$(\sqrt{2})^n = 64$

$2^{\frac{1}{2}n} = 64$

$\frac{1}{2}n = 6$

$n = 12$

2008

Q1 – 4 marks

The first term of an arithmetic sequence is 2 and the 20th term is 97. Obtain the sum of the first 50 terms. **4**

Written Solutions

$u_n = a + (n-1)d$

First term , $a = 2$

20th term $u_{20} = a + (20-1)d$

$97 = 2 + 19d$

$d = 5$

$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$S_{50} = \frac{50}{2}\left[2(2) + (50-1)\times 5\right]$

$= 6225$

2007

Q9 – 5 marks

Show that $\sum_{r=1}^{n}(4-6r) = n-3n^2$. **2**

Hence write down a formula for $\sum_{r=1}^{2q}(4-6r)$. **1**

Show that $\sum_{r=q+1}^{2q}(4-6r) = q-9q^2$. **2**

Written Solutions

$$\sum_{r=1}^{n}(4-6r) = n-3n^2$$

$$\sum_{1}^{n} 1 = n$$

$$\sum_{1}^{n} r = \frac{1}{2}n(n+1)$$

$$\sum_{1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

$$\text{LHS} = 4\sum_{r=1}^{n}1 - 6\sum_{r=1}^{n}r$$

$$= 4n - 6\cdot\frac{1}{2}n(n+1)$$

$$= 4n - 3n^2 - 3n$$

$$= n - 3n^2 = \underline{\underline{\text{RHS}}}$$

$$\sum_{r=1}^{2q}(4-6r) = 2q - 3(2q)^2$$

$$= \underline{\underline{2q - 12q^2}}$$

$$\sum_{r=q+1}^{2q}(4-6r) = \sum_{r=1}^{2q}4-6r - \sum_{r=1}^{q}4-6r$$

$$= 2q - 12q^2 - (q - 3q^2)$$

$$= \underline{\underline{q - 9q^2}}$$ as ~~req~~ req'd