Sequences and Series

2013

Q17 – 10 marks

Write down the sums to infinity of the geometric series

$$1 + x + x^2 + x^3 + ...$$

and

$$1 - x + x^2 - x^3 + ...$$

valid for $|x| < 1$.

Assuming that it is permitted to integrate an infinite series term by term, show that, for $|x| < 1$,

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + ...\right). \quad \mathbf{7}$$

Show how this series can be used to evaluate ln 2.

Hence determine the value of ln 2 correct to 3 decimal places. **3**

Marking Instructions

Write down the sums to infinity of the geometric series $1 + x + x^2 + x^3 + \ldots$ **and** $1 - x + x^2 - x^3 + \ldots$ **Valid for** $\lvert x \rvert < 1$. **Assuming that it is permitted to integrate an infinite series term by term, show that, for** $\lvert x \rvert < 1$, $\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + ...\right)$	7	
Show how this series can be used to evaluate ln 2. **Hence determine the value of ln 2 correct to 3 decimal places.** $1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$	3	•[1] Correct statement of sum.

Integrating the first of these gives: $x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5+\ldots=-\ln(1-x)+c$	•3 Correct integration of both sides.[1]
Putting $x=0$ gives $c=0$.	•4 Correct evaluation of c.[3]
Similarly, $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4\ldots=\ln(1+x)$	•5 Correct integration of both sides.[1]
Adding together gives: $2\left(x+\frac{1}{3}x^3+\frac{1}{5}x^5+\ldots\right)=\ln(1+x)-\ln(1-x)$	•6 Evidence of appropriate method.
$\left[=\ln\frac{1+x}{1-x}\right]$ as required.	•7 Appropriate intermediate step.
OR $2+2x^2+2x^4+\cdots = \frac{1}{1+x}+\frac{1}{1-x}$	•3 Adds series.
$\therefore\ 2x+\frac{2}{3}x^3+\frac{2}{5}x^5+\cdots$	•4 Integrates LHS

$=\ln(1+x)\ldots$ $\ldots-\ln(1-x)+c$	•5 Integrates $\ln(1+x)$ •6 Integrates $\ln(1-x)$

Putting $x=0$ gives $c=0$. $\left[=\ln\frac{1+x}{1-x}\right]$ as required. **OR**	•7 Correct evaluation of c.[1,3]

$f(x)=ln\left(\frac{1+x}{1-x}\right)$	$f(0)=0$	•3 Evidence of appropriate use of Maclaurin.[5,7]
$f'(x)=2(1-x^2)^{-1}$ or equivalent	$f'(0)=2$	•4 All five derivatives correct OR first two derivatives *and* first three evaluations correct.[5]
$f''(x)=4x(1-x^2)^{-2}$	$f''(0)=0$	
$f'''(x)=16x^2(1-x^2)^{-3}+4(1-x^2)^{-2}$	$f'''(0)=4$	•5 All six evaluations correct OR final three derivatives correct *and* final three evaluations correct.[5]
$f^{IV}(x)=96x^3(1-x^2)^{-4}+48x(1-x^2)^{-3}$	$f^{IV}(0)=0$	
$f^{V}(x)=768x^4(1-x^2)^{-5}+576x^2(1-x^2)^{-4}+48(1-x^2)^{-3}$	$f^{V}(0)=48$	
$\therefore f(x)=0+2.1x+0x^2+\frac{4}{3!}x^3+0x^4+\frac{48}{5!}x^5+\cdots$		•6 Correctly substitutes obtained values into Maclaurin.
$=2x+\frac{2}{3}x^3+\frac{2}{5}x^5+\cdots$		•7 Simplification *en route* to required result.[8]
so $f(x)=\ln\left(\frac{1+x}{1-x}\right)=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right)$ as required.		

Now choose x such that $=\frac{1+x}{1-x}=2$,	•8 States appropriate equation.
ie $1+x=2-2x$, so $x=\frac{1}{3}$	•9 Correctly solves equation.[4]
So $\ln 2=2\left(\frac{1}{3}+\frac{1}{81}+\frac{1}{1215}+\frac{1}{15309}+\ldots\right)$	
$=0{\cdot}693$ to 3 d.p.	•10 Obtains accurate approximation.[2,6]

2012

Q2 – 5 marks

The first and fourth terms of a geometric series are 2048 and 256 respectively. Calculate the value of the common ratio. **2**

Given that the sum of the first n terms is 4088, find the value of n. **3**

Marking Instructions

$a = 2048$ and $ar^3 = 256$ $\Rightarrow r^3 = \frac{1}{8}$	**1M**	valid approach
$\Rightarrow r = \frac{1}{2}$.	**1**	correct answer only, 2 marks
$S_n = \dfrac{a(1 - r^n)}{1 - r}$ $\Rightarrow \dfrac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = \dfrac{4088}{2048} = \dfrac{511}{256}$	**1M**	for sum formula
$\Rightarrow 1 - \left(\frac{1}{2}\right)^n = \dfrac{511}{256} \times \dfrac{1}{2} = \dfrac{511}{512}$		
$\dfrac{1}{2^n} = 1 - \dfrac{511}{512} = \dfrac{1}{512}$	**1**	
$\Rightarrow 2^n = 512 \Rightarrow n = 9$	**1**	any valid method

2011

Q8 – 4 marks

Write down an expression for $\displaystyle\sum_{r=1}^{n} r^3 - \left(\sum_{r=1}^{n} r\right)^2$ **1**

and an expression for

$$\sum_{r=1}^{n} r^3 + \left(\sum_{r=1}^{n} r\right)^2.$$ **3**

Marking Instructions

$\displaystyle\sum_{r=1}^{n} r^3 - \left(\sum_{r=1}^{n} r\right)^2 = \frac{n^2(n+1)^2}{4} - \left(\frac{n(n+1)}{2}\right)^2 = 0$	**1**
$\displaystyle\sum_{r=1}^{n} r^3 + \left(\sum_{r=1}^{n} r\right)^2 = \frac{n^2(n+1)^2}{4} + \left(\frac{n(n+1)}{2}\right)^2$	**1**
$= \dfrac{n^2(n+1)^2}{4} + \dfrac{n^2(n+1)^2}{4}$	**1**
$= \dfrac{n^2(n+1)^2}{2}$	**1**

2011

Q13 – 9 marks

The first three terms of an arithmetic sequence are $a, \frac{1}{a}, 1$ where $a < 0$.

Obtain the value of a and the common difference. **5**

Obtain the smallest value of n for which the sum of the first n terms is greater than 1000. **4**

Marking Instructions

Method 1

Let d be the common difference. Then

$u_3 = 1 = a + 2d$ and $u_2 = \frac{1}{a} = a + d$ **1**

$1 = a + 2\left(\frac{1}{a} - a\right)$ **1**

$a = a^2 + 2 - 2a^2$

$a^2 + a - 2 = 0$ **1**

$(a+2)(a-1) = 0 \Rightarrow a = -2$ since $a < 0$. **1**

$a = -2$ gives $2d = 3$ and hence $d = \frac{3}{2}$. **1**

Method 2

$u_1 = a, u_2 = \frac{1}{a}, u_3 = 1$ **M1**

$\Rightarrow \frac{1}{a} - a = 1 - \frac{1}{a}$ **1**

$\Rightarrow 1 - a^2 = a - 1$

$\Rightarrow a^2 + a - 2 = 0$ **1**

$(a+2)(a-1) = 0 \Rightarrow a = -2$ since $a < 0$. **1**

$d = u_3 - u_2 = 1 - \frac{1}{a} = \frac{3}{2}$ **1**

$S_n = \frac{n}{2}[2a + (n-1)d]$

$= \frac{n}{2}\left[-4 + \frac{3}{2}n - \frac{3}{2}\right]$ **1**

$= \frac{1}{4}[3n^2 - 11n]$

$\therefore\ 3n^2 - 11n > 4000$ **1**

$n^2 - \frac{11}{3}n > \frac{4000}{3}$

$\left(n - \frac{11}{6}\right)^2 > \frac{48000}{36} + \frac{121}{36} = \frac{48121}{36}$

$n - \frac{11}{6} > \frac{\sqrt{48121}}{6}$

$n > \frac{\sqrt{48121} + 11}{6} \approx 38.39$

So the least value of n is 39. **1** for value

1 for suitable justification

2010

Q2 – 5 marks

The second and third terms of a geometric series are –6 and 3 respectively.

Explain why the series has a sum to infinity, and obtain this sum. **5**

Marking Instructions

Let the first term be a and the common ratio be r. Then		
$ar = -6 \quad \text{and} \quad ar^2 = 3$	**1**	{both terms needed}
Hence		
$r = \frac{ar^2}{ar} = \frac{3}{-6} = -\frac{1}{2}.$	**1**	evaluating r
So, since $\lvert r \rvert < 1$, the sum to infinity exists.	**1**	justification
$S = \frac{a}{1-r}$	**1**	correct formula
$= \frac{12}{1-(-\frac{1}{2})} = \frac{12}{\frac{3}{2}}$		
$= 8.$	**1**	the sum to infinity

2009

Q12 – 6 marks

The first two terms of a geometric sequence are $a_1 = p$ and $a_2 = p^2$. Obtain expressions for S_n and S_{2n} in terms of p, where $S_k = \sum_{j=1}^{k} a_j$. **1,1**

Given that $S_{2n} = 65S_n$ show that $p^n = 64$. **2**

Given also that $a_3 = 2p$ and that $p > 0$, obtain the exact value of p and hence the value of n. **1,1**

Marking Instructions

$$a_j = p^j \Rightarrow S_k = p + p^2 + \dots + p^k = \frac{p(p^k - 1)}{p - 1}$$

$$S_n = \frac{p(p^n - 1)}{p - 1} \qquad \mathbf{1}$$

$$S_{2n} = \frac{p(p^{2n} - 1)}{p - 1} \qquad \mathbf{1}$$

$$\frac{p(p^{2n} - 1)}{p - 1} = \frac{65p(p^n - 1)}{p - 1}$$

$$(p^n + 1)(p^n - 1) = 65(p^n - 1) \qquad \mathbf{1}$$

$$p^n + 1 = 65 \qquad \mathbf{1}$$

$$\Rightarrow p^n = 64$$

$$a_2 = p^2 \Rightarrow a_3 = p^3 \text{ but } a_3 = 2p \text{ so } p^3 = 2p$$

$$\Rightarrow p^2 = 2 \Rightarrow p = \sqrt{2} \text{ since } p > 0. \qquad \mathbf{1}$$

$$p^n = 64 = 2^6 = (\sqrt{2})^{12}$$

$$n = 12 \qquad \mathbf{1}$$

2008

Q1 – 4 marks

The first term of an arithmetic sequence is 2 and the 20th term is 97. Obtain the sum of the first 50 terms. **4**

Marking Instructions

Let the common difference be d. General term is $a + (n - 1)d$.	1
So $2 + 19d = 97 \Rightarrow d = 5$.	1
Sum of an arithmetic series is $\frac{n}{2}[2a + (n - 1)d]$.	1
Required sum is $\frac{50}{2}\{4 + 49 \times 5\} = 6225$.	1

2007

Q9 – 5 marks

Show that $\sum_{r=1}^{n}(4-6r) = n - 3n^2$. **2**

Hence write down a formula for $\sum_{r=1}^{2q}(4-6r)$. **1**

Show that $\sum_{r=q+1}^{2q}(4-6r) = q - 9q^2$. **2**

Marking Instructions

$\sum_{r=1}^{n}(4-6r) = 4\sum_{r=1}^{n} - 6\sum_{r=1}^{n} r$	1M
$= 4n - 3n(n+1)$	1
$= n - 3n^2$	
$\sum_{r=1}^{2q}(4-6r) = 2q - 12q^2$	1
$\sum_{r=q+1}^{2q}(4-6r) = \sum_{r=1}^{2q}(4-6r) - \sum_{r=1}^{q}(4-6r)$	1M
$= (2q - 12q^2) - (q - 3q^2)$	1
$= q - 9q^2$.	

Arithmetic Series could be used, so, for the first two marks:

$a = -2, d = -6 \Rightarrow S_n = \frac{n}{2}\{2(-2) + (n-1)(-6)\}$	1
$= -2n - 3n^2 + 3n = n - 3n^2$	1