Further Differentiation

2013

Q11 – 6 marks

A curve has equation

$$x^2 + 4xy + y^2 + 11 = 0.$$

$\therefore$ implicit eq^n

Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(-2, 3)$.

Written Solutions

$$x^2 + 4xy + y^2 + 11 = 0$$

$$2x + 4y + 4x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \qquad *$$

When $x = -2$, $y = 3$

$$-4 + 12 - 8\frac{dy}{dx} + 6\frac{dy}{dx} = 0$$

$$-2\frac{dy}{dx} = -8$$

$$\frac{dy}{dx} = 4$$

$$*' \quad 2 + 4\frac{dy}{dx} + 4\frac{dy}{dx} + 4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{dy}{dx} + 2y\frac{d^2y}{dx^2} = 0$$

When $x = -2$, $y = 3$, $\frac{dy}{dx} = 4$

$$2 + 16 + 16 + (-8)\frac{d^2y}{dx^2} + 32 + 6\frac{d^2y}{dx^2} = 0$$

$$-2\frac{d^2y}{dx^2} = -66$$

$$\frac{d^2y}{dx^2} = 33$$

2012

Q13 – 10 marks

A curve is defined parametrically, for all t, by the equations

$$x = 2t + \frac{1}{2}t^2, \qquad y = \frac{1}{3}t^3 - 3t.$$

Obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ as functions of t.

Find the values of t at which the curve has stationary points and determine their nature.

Show that the curve has exactly two points of inflexion.

Written Solutions

$$x = 2t + \frac{1}{2}t^2 \qquad y = \frac{1}{3}t^3 - 3t$$

$$\frac{dx}{dt} = 2 + t \qquad \frac{dy}{dt} = t^2 - 3$$

So, $$\frac{dy}{dx} = \frac{t^2 - 3}{2 + t}$$

From above, $$\frac{d^2x}{dt^2} = 1 \quad \text{and} \quad \frac{d^2y}{dt^2} = 2t$$

So, $$\frac{d^2y}{dx^2} = \frac{x'y'' - x''y'}{(x')^3}$$

$$= \frac{(2+t)(2t) - (1)(t^2-3)}{(2+t)^3}$$

$$= \underline{\underline{\frac{(t+1)(t+3)}{(2+t)^3}}}$$

$\frac{dy}{dx} = 0$ @ S.P.

So, $\frac{t^2-3}{2+t} = 0 \Rightarrow t^2 - 3 = 0 \Rightarrow \underline{\underline{t = \pm\sqrt{3}}}$

<u>Nature</u>

when $t = \sqrt{3}$, $\frac{d^2y}{dx^2} > 0 \Rightarrow$ minimum

when $t = -\sqrt{3}$, $\frac{d^2y}{dx^2} < 0 \Rightarrow$ maximum

PTO

For points of inflexion $\frac{d^2y}{dx^2} = 0$, so, $\frac{(t+1)(t+3)}{(t+2)^3} = 0$

This is just a quadratic and has two roots, $\underline{\underline{t=-1}}$ and $\underline{\underline{t=-3}}$.

2011

Q3a – 3 marks

Obtain $\frac{dy}{dx}$ when y is defined as a function of x by the equation

$$y + e^y = x^2.$$

Written Solutions

∴ implicit eqⁿ

$$y + e^y = x^2$$

$$\frac{dy}{dx} + e^y \cdot \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx}(1 + e^y) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{(1+e^y)}$$

2011

Q7 – 4 marks

A curve is defined by the equation $y = \dfrac{e^{\sin x}(2+x)^3}{\sqrt{1-x}}$ for $x < 1$.

Calculate the gradient of the curve when $x = 0$.

Written Solutions

see "Differentiation" PPQ's for quotient rule sol^n

$$y = \frac{e^{\sin x}(2+x)^3}{\sqrt{1-x}}$$ ← implicit diff

$$y\sqrt{1-x} = e^{\sin x}(2+x)^3$$

$$y(1-x)^{\frac{1}{2}} = e^{\sin x}(2+x)^3$$

$$(1-x)^{\frac{1}{2}}\frac{dy}{dx} + y\cdot\frac{1}{2}(1-x)^{-\frac{1}{2}}\cdot(-1) = \cos x\, e^{\sin x}\cdot(2+x)^3 + 3(2+x)^2 e^{\sin x} \quad *$$

When $x=0$, $y = \dfrac{e^0 2^3}{1} = 8$

Using $*$ this leads to $\dfrac{dy}{dx} = 24$

<u>ALTERNATIVE METHOD</u> ← log diff^n

$$y = \frac{e^{\sin x}(2+x)^3}{\sqrt{1-x}}$$

2010

Q13 – 10 marks

Given $y = t^3 - \frac{5}{2}t^2$ and $x = \sqrt{t}$ for $t > 0$, use parametric differentiation to express $\frac{dy}{dx}$ in terms of t in simplified form.

Show that $\frac{d^2y}{dx^2} = at^2 + bt$, determining the values of the constants a and b.

Obtain an equation for the tangent to the curve which passes through the point of inflexion.

Written Solutions

$$x = \sqrt{t} \qquad\qquad y = t^3 - \frac{5}{2}t^2$$

$$\frac{dx}{dt} = \frac{1}{2}t^{-\frac{1}{2}} \qquad\qquad \frac{dy}{dt} = 3t^2 - 5t$$

$$= \frac{1}{2\sqrt{t}}$$

$$\frac{dy}{dx} = \frac{3t^2 - 5t}{\left(\frac{1}{2\sqrt{t}}\right)}$$

$$= 2\sqrt{t} \cdot t(3t-5)$$

$$= 6t^{5/2} - 10t^{3/2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{6 \times \frac{5}{2}t^{3/2} - 10 \times \frac{3}{2}t^{1/2}}{\frac{1}{2}t^{-1/2}}$$

$$= 30t^2 - 30t$$

i.e. $\underline{\underline{a = 30}}$, $\underline{\underline{b = -30}}$

@ the point of inflexion, $\frac{d^2y}{dx^2} = 0 \Rightarrow 30t^2 - 30t = 0$

$\Rightarrow t = 0$ or 1

But $t > 0 \Rightarrow t = 1 \Rightarrow \frac{dy}{dx} = -4$

and the point is $(1, -3/2)$

Hence the tangent is $y + \frac{3}{2} = -4(x - 1)$

i.e. $2y + 8x = 5$

2009

Q1b – 7 marks

(*a*) Given $f(x) = (x+1)(x-2)^3$, obtain the values of x for which $f'(x) = 0$.

(*b*) Calculate the gradient of the curve defined by $\frac{x^2}{y} + x = y - 5$ at the point $(3, -1)$.

Written Solutions

(b) $x^2 + xy = y^2 - 5y$

$2x + x\frac{dy}{dx} + y = 2y\frac{dy}{dx} - 5\frac{dy}{dx}$

when $x = 3$, $y = -1$

$6 + 3\frac{dy}{dx} - 1 = -2\frac{dy}{dx} - 5\frac{dy}{dx}$

$-10\frac{dy}{dx} = 5$

$\frac{dy}{dx} = -\frac{1}{2}$

2009

Q11 – 5 marks

The curve $y = x^{2x^2+1}$ is defined for $x > 0$. Obtain the values of y and $\frac{dy}{dx}$ at the point where $x = 1$.

Written Solutions

$$y = x^{2x^2+1}$$

$\therefore$ when $x = 1$, $y = 1$

$$\Rightarrow \quad \ln y = \ln\left(x^{2x^2+1}\right)$$

$$\ln y = (2x^2+1)\ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x^2+1}{x} + 4x\ln x$$

Hence, when $x = 1$ and $y = 1$, $\frac{dy}{dx} = \left(\frac{2x^2+1}{x} + 4x\ln x\right) \times y$

$$= \underline{\underline{3}}$$

2008

Q2b – 3 marks

(*a*) Differentiate $f(x) = \cos^{-1}(3x)$ where $-\frac{1}{3} < x < \frac{1}{3}$.

(*b*) Given $x = 2\sec\theta$, $y = 3\sin\theta$, use parametric differentiation to find $\frac{dy}{dx}$ in terms of θ.

Written Solutions

$x = 2\sec\theta \qquad y = 3\sin\theta$

$\frac{dx}{d\theta} = 2\sec\theta\tan\theta \qquad \frac{dy}{d\theta} = 3\cos\theta$

$$\frac{dy}{dx} = \frac{3\cos\theta}{2\sec\theta\tan\theta}$$

$$= \frac{3\cos^3\theta}{2\sin\theta}$$

2008

Q5 – 6 marks

A curve is defined by the equation $xy^2 + 3x^2y = 4$ for $x > 0$ and $y > 0$.

Use implicit differentiation to find $\frac{dy}{dx}$.

Hence find an equation of the tangent to the curve where $x = 1$.

Written Solutions

$$xy^2 + 3x^2y = 4$$

$$y^2 + 2xy\frac{dy}{dx} + 6xy + 3x^2\frac{dy}{dx} = 0$$

$$(2xy + 3x^2)\frac{dy}{dx} = -y^2 - 6xy$$

$$\frac{dy}{dx} = \frac{-y^2 - 6xy}{2xy + 3x^2}$$

When $x = 1$

$$y^2 + 3y = 4$$

$$y^2 + 3y - 4 = 0$$

$$(y+4)(y-1) = 0$$

So, $y = -4$ or 1. $y = 1$ since $y > 0$

Hence $(1, 1)$ $\quad \frac{dy}{dx} = -\frac{7}{5}$

and tangent is $\quad y - 1 = -\frac{7}{5}(x - 1)$

$$5y + \underline{\underline{7x - 12}} = 0$$

2007

Q2b – 3 marks

Obtain the derivative of each of the following functions:

(*a*) $f(x) = \exp(\sin 2x)$;

(*b*) $y = 4^{(x^2+1)}$.

Written Solutions

(b)

$$y = 4^{x^2+1}$$

$$\ln y = \ln\left(4^{(x^2+1)}\right)$$

$$\ln y = (x^2+1)\ln 4$$

$$\frac{1}{y}\frac{dy}{dx} = 2x \ln 4$$

$$\frac{dy}{dx} = 2x \ln 4 \cdot y$$

$$= \underline{\underline{2x \ln 4 \cdot 4^{(x^2+1)}}}$$